POLY-UNIVERSE IN TEACHER EDUCATION

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METHODOLOGICAL STUDY

Handbook for pre-service and in-service teachers and students

Poly-Universe in Teacher Training Education

Erasmus+ PUNTE Project: 2020-1-HU01-KA203-078810

Project Coordinator: Eszterházy Károly Catholic University, Hungary

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INTRODUCTION

The Poly-Universe in Teacher Training Education (PUNTE) project was launched in September 2020, supported by the European Commission Strategic Partnerships Framework. The aim of the project is to develop innovative, transdisciplinary pedagogical methods that are primarily related to a revolutionary educational tool, called Poly-Universe. With this PUNTE Methodological Handbook, the reader obtains a study volume that presents the possibilities of introducing and applying the Poly-Universe kit in teacher training, in the spirit of striving for completeness.

The birth of the Poly-Universe toolkit itself is the result of a fundamentally art-driven concept, which, however, in addition to aesthetic aspects, opened up new perspectives first in the teaching of mathematics and then in further disciplines. Today, we can declare that the Poly-Universe kit has become part of everyday life in many primary and secondary schools. The aim of this volume is to facilitate the incorporation of this everyday practice into teacher training higher education and the transfer of methodological foundations to teachers and teacher candidates.

The volume consists of four major and two minor chapters. In the first chapter, the reader can become acquainted with the teaching-learning strategies and theories that provide a solid foundation for the practical implementation of the Poly-Universe. From inquiry-based learning theory to the study of spatial abilities, from basic problem-solving theories to psychological principles related to the learning process, such as engagement and motivation, this chapter discusses various aspects of the required theoretical background.

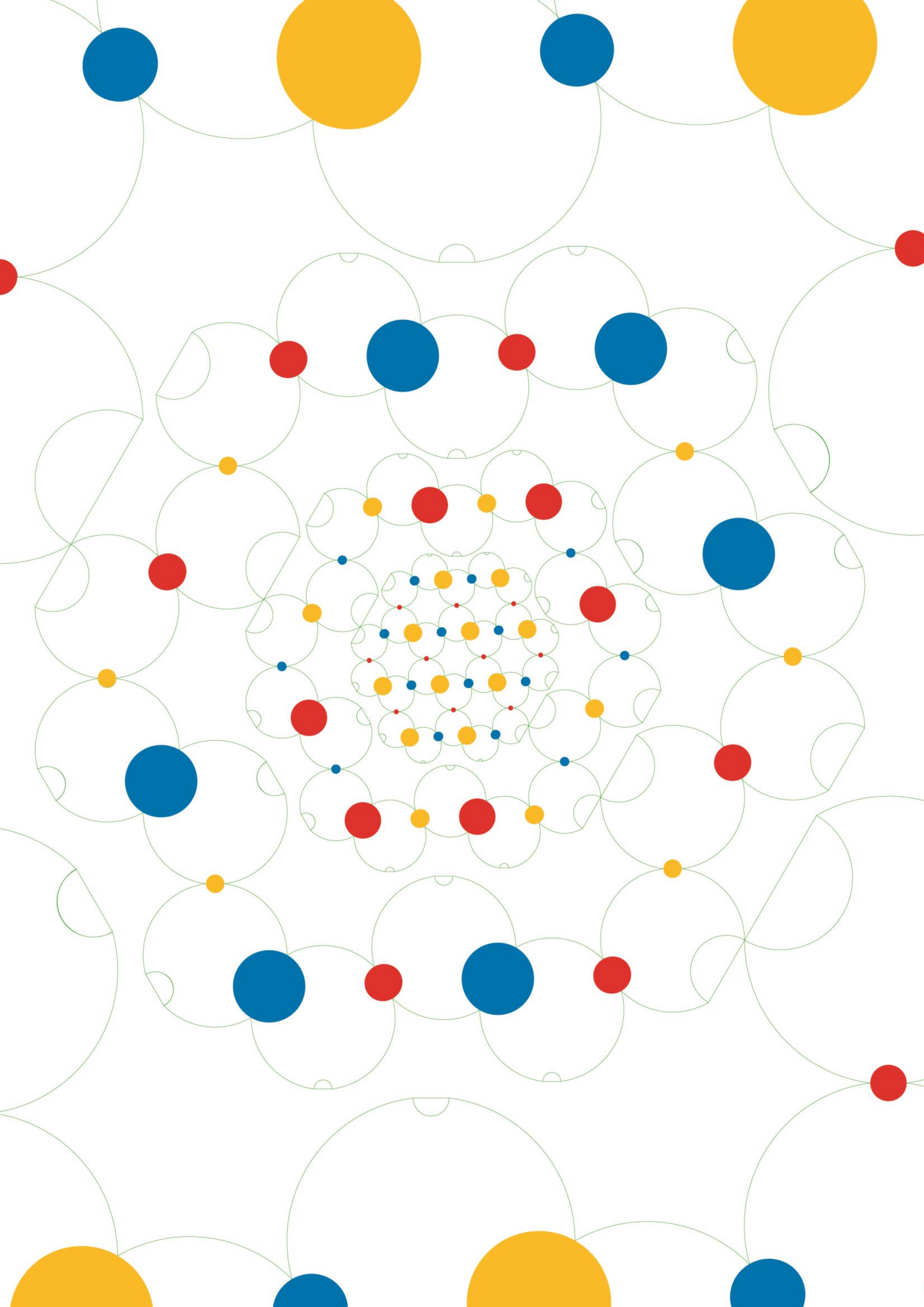
Going back to the roots of art-based education and learning, the second chapter shows how an art-oriented approach to symmetry, dimension, and transformation can add an incredible plus to purely geometric, math-based education. To understand this, this chapter takes the reader back to the beginnings of the concrete and constructive art trends of the 20th century, but this section also discusses in detail the impact of this historical overview to present educational approaches, and the artistic power and possibilities that appear specifically in the use of the Poly-Universe.

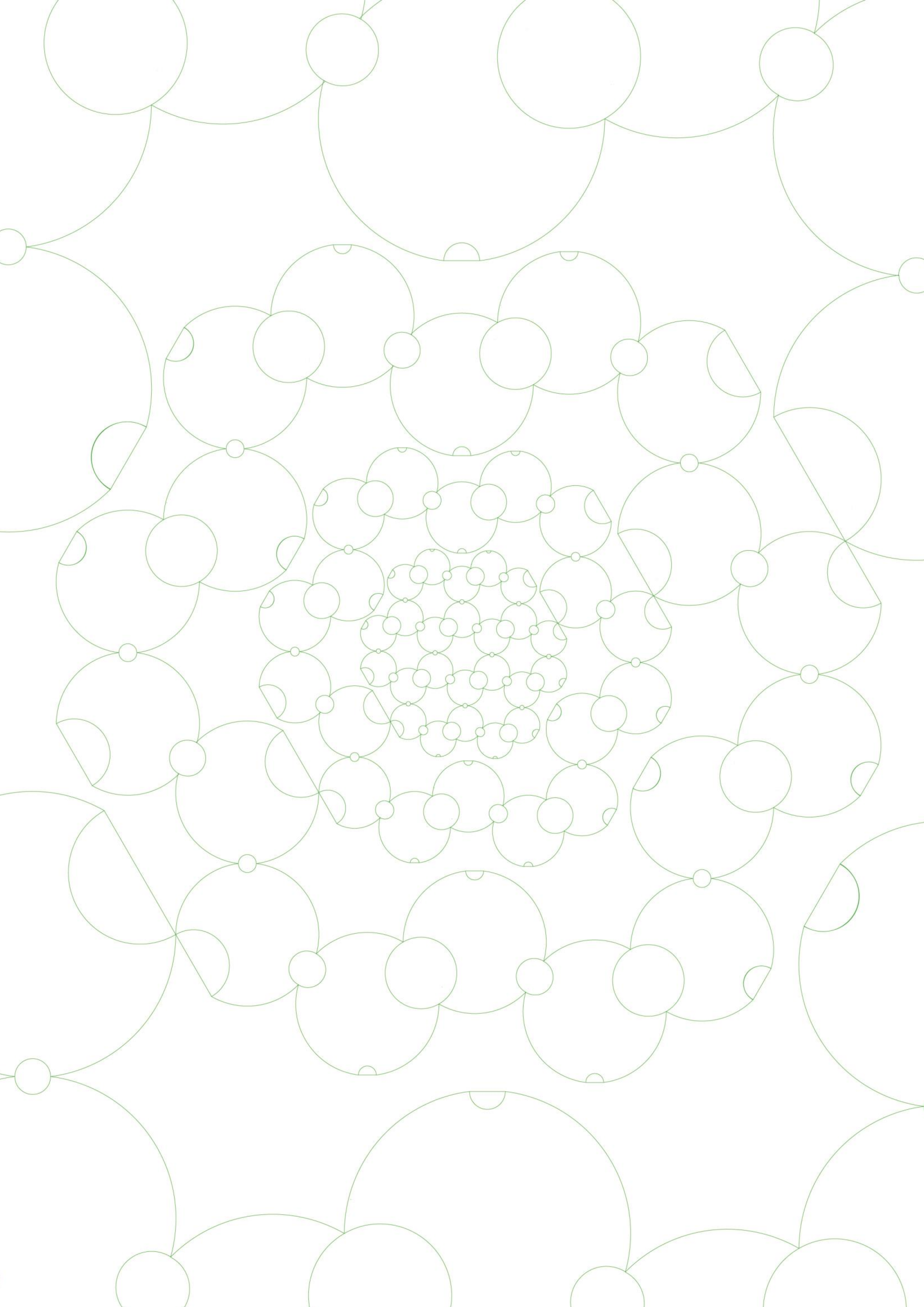
The third chapter provides an insight into the methodological foundations. Cooperative learning strategies, different manifestations of creativity, or holistic approaches all present outstanding possibilities for the methodologically sound use of the Poly-Universe.

The remaining chapters provide an approach to the methodology of the Poly-Universe kit from the practical side. Here we get an answer to where, how and in what context we can and should use the tool in the teaching-learning process. Good practices extend beyond mathematics to other disciplines, further reinforcing the transdisciplinary nature of Poly-Universe-based teaching. Finally, in the sixth chapter – in line with the needs of the 21st century – we can read about the methodological and curricular aspects of the adaptation of the Poly-Universe tool to the electronic learning environment.

Overall, this volume presents us with a broad horizon of methodological and theoretical aspects of the Poly-Universe tool. The authors and the participants in the PUNTE project hope that the book will provide the reader with a solid theoretical basis and practical knowledge instantly applicable to teacher training education.

Eger, October 31, 2021





I THEORETICAL BACKGROUND — TEACHING-LEARNING THEORIES BEHIND THE METHOD

1.1 Constructivism

Constructivism is one of the most influential theories of learning and intellectual development of human kind. It is based on its three roots: Problem solving, misconceptions, critical barriers, and epistemological obstacles, and Theories of cognitive development. The main motivation for constructivist researchers is their need to help students to overcome learning difficulties of mathematics concerning: students' poor understanding of concepts, over-developed procedures and students' difficulties with recalling knowledge and transferring it to new tasks. The focus of constructivist research of students' rezoning is on the students' strategies and approaches. (Confrey and Kazak, 2006)

Active participation of the learners plays a crucial role in this theory. According to constructivism: 'Knowledge is constructed in the mind of the learner.' (Bodner, 1986, p.873). During the learning process the learners create their knowledge in their specific way and do not passively get it from the outside. Through exploring their learning environment and interacting with the real physical world, the learners gain a lot of experience. Construction of new knowledge starts when a learner exploring his/her learning environment, recognizes a problem and decides to solve it.

Then the learner begins to act in order to eliminate that perturbation or to overcome the feeling of imbalance. (Confrey and Kazak, 2006) The learner makes comparisons and in many cases transforms the primary situation. How successful the act is, depends on the learner's amount of active participation. (Sinclair, 1987) When the learner gets the results of the action he/she often uses some representations for their presentation. After all, the student assesses whether the problem has been solved or whether more action is needed. It is usually necessary to repeat the whole procedure several times before the student can solve the problem. Through this circular process the learner creates new ideas, modifies or rejects the existing ones. (Confrey and Kazak, 2006) While the creation of knowledge is an individual activity; the learners construct it through interaction and collaboration with their teachers, classmates and other people. As different people have different experiences, knowledge, learning styles etc, they look at things from different points of view. When the learners explain their thoughts and discuss them with one another they face the points of view of other people. It can help learners to get a deep understanding of the problem he/she is dealing with and to modify his/her ideas or to build more and more abstract ones. Finally the learner connects new ideas and information with his/her existing knowledge. Then, through the processes of assimilation and accommodation the learner acquires his/her knowledge. Over time, after a lot of interactions of the learning process, the learner's knowledge becomes more and more complex and more sophisticated (Naylor and Keogh, 1999; Taber, 2011; Sjoberg, 2010; Iran-Nejad, 1995).

It should be pointed out that the learners in collaboration with their classmates conduct their own constructivist learning process and are responsible for it.

The role of teachers is crucially important in constructivist learning theory. Teachers develop and prepare a variety of manipulatives and tools, and interactive learning materials. They can also use new technologies to imitate concrete situations of real life. Teachers create, arrange and structure the learning environment in order to facilitate learning for their students. Teachers guide the students' learning process. They encourage students to explore and analyze the prepared learning environment. They urge students to use tools and manipulatives and to collaborate with other students. They encourage students' dialogs and require students to ask the teacher and their classmates different questions. Teachers suggest that the students elaborate their initial responses. In order to help their students to realize that some of their ideas are not correct, teachers arrange situations from which the students can gain experiences that point out the contradictions to students' knowledge. (Tobin and Tippins, 1993; Brooks and Brooks, 1993; Doglu and Kalender, 2007). According to Good and Brophy (1994) the best social interaction among students occurs when learning in small groups. Therefore, the teachers organize learning in small groups for their students and on the bases of constructivism, two learning approaches have developed: collaborative and cooperative learning.

Poly-Universe is an ideal manipulative tool, the teachers can create a very rich and varied learning environment by utilizing it and can organize learning for their students according to constructivist learning theory.

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1.2 Problem Solving in Mathematics Teaching. Bruner's Representation Theory

The general development of problem-solving skills is an important goal in mathematics teaching. The problem-solving process involves the development and use of several aspects of the students' cognitive skills, reasoning being one of the most important ones.

With the help of modern imaging techniques (fMRI) O'Boyle from Texas Tech University found that during problem solving the 13 year old S.M., a student gifted in mathematics, used approximately five-six times more areas of the brain than students with average skills (Randall, 2012).

One might raise the question of whether problem solving is teachable at all, and if it is how to teach it effectively. However, we pedagogues believe in teaching problem-solving skills, and that we can help students of different ages and different skills through familiarizing them with different strategies.

Modelling realistic mathematical situations is also important, as well as familiarizing students with unsolvable problems. In addition, there is a need for problems containing redundant information as considered in the question. Teaching these problems in a colorful and varied way is crucial. Bruner already discussed different representations some 60 years ago. Methods should be adapted to age level and consideration given to Bruner's stages of representation (Bruner, 1974).

1.2.1 Bruner's learning theory

Bruner's interests centred on development, cognitive processes and conceptualization. He holds that the cognitive development of children is influenced by experiences, education and upbringing.

The main ideas of his theory can be summarized as follows:

1. *Teaching a subject has to be based on the fundamental principles (patterns) of the given science.* In teaching mathematics, one has to focus on basic principles which make possible an economical description and explanation of reality and the world of mathematics.
2. *Basic principles of all subjects can be taught to all children, regardless of age and social environment, in a simple, adequate way, with the help of representation models understandable to them, building on their current cognitive capabilities.*

Bruner lays an emphasis on the teaching of basic patterns (structures). He points out the following advantages of teaching fundamental principles:

1. Students can grasp the material easier if they understand the basics.
2. Specific details are easily forgotten if the information is not structured.
3. Understanding basic concepts also advances transfer.
4. Students encounter few difficulties when moving on to the next level.

According to Bruner, in order for students to understand the content, an early stage of intuitive thinking is essential and the subject matter should be revisited later, taking into consideration their stage of cognitive development. This approach is called the *spiral* curriculum.

Bruner's guidelines for teaching mathematics are as follows:

- All concepts should be introduced at a young age.
- Teaching mathematics should be based on the current state of the field.
- The subject matter should be presented in a way that makes it most readily understandable by the learner.

The last two points often conflict in the educational process. This is one of the reasons why Bruner's approach has received considerable criticism.

1.2.2 Bruner's representation theory

Bruner was concerned with how knowledge is represented and encoded in memory. According to his theory, all cognitive processes can take place on three conceptual levels, and knowledge is represented in three different modes.

1. *Enactive representation*
   Knowledge acquisition, achieving a goal is carried out through concrete physical actions, manipulation of objects.
2. *Iconic representation*
   Knowledge is acquired with the help of images, pictures and imaginary situations, for example, tree diagrams or geometric representation of algebraic problems.
3. *Symbolic representation*
   At this stage, knowledge is acquired through mathematical symbols and language.

These three modes of representation are instrumental at each stage of the educational process. Progression from one mode to another enhances flexibility and the efficiency of problem-solving. The iconic representation plays an important role all through mathematics teaching.

In the *enactive stage,* learning is characterized by concrete, hands-on actions and manipulations-the younger the students, the more concrete actions they need.

The *iconic stage* involves the use of images and other visuals, as well as imaginary situations.

The effectiveness of the learning process can be increased by consciously varying the modes of representation. The visualization of a problem can facilitate a better understanding. Students should be encouraged with patience and extensive practice to use visual representation.

'Using concrete and iconic representations is necessary not only for the so-called slow students or elementary students. These representations are important for all students and are useful throughout the entire learning process' (Wittmann, 1998).

The *symbolic stage* involves using words and mathematical symbols. The right timing for introducing the symbolic stage is crucial since students in the classroom are not on the same level.

The learning process can be made more efficient if the modes of representation are consciously varied.

Minimal teacher guidance, as a means of exploratory learning, is not enough for efficient learning. The other extreme, maximum teacher guidance or frontal teaching, is also not too efficient (Kirschner et *al.*, 2006). Students need to gain as much experience as possible through individual work. However, if they do not receive any help at all or too little help they might not advance. Teachers should help but not in excess. They should help in such a measure that students are left with a reasonable proportion of work (Pólya, 2000).

How do we solve word problems? There are different models for this. We propose Pólya's model for solving problems in class. Pólya puts forth four principles for solving problems and developing cognitive skills (Pólya, 2000).

1. *Understand the problem!*
2. *Look for connections between data and the unknown! If you cannot find direct connections, look for helping problems! Make a plan for solving the problem!*
3. *Carry out the plan!*
4. *Check the solution!*

The first step, understanding the problem, points out the importance of carefully reading and systematizing data. The question and the constraint jointly determine the next step, i.e., making a plan. Conscious problem solving always follows a plan. In this process students can draw on their current knowledge. This is the most fully described stage. In this stage students have to look for connections between the data and the unknown. If students find the problem easy, it means that they have already solved a number of similar problems. This means that we no longer speak of cognitive development but of knowledge automation. When dealing with a more difficult problem, students should be encouraged not to give up. Any correctly-solved sub-problem, or problems solved in a similar way may prove helpful. It is essential for students to learn to think, to ask questions related to the problem, and to become familiar with a variety of problems in a realistic way. These types of problems develop students' skills, they become more active and increase their mathematics self-concept.

In the third stage, when the plan is carried out, it is important to do the operations correctly.

The fourth step, checking the solution, is one of the most important steps. If the problem is solved correctly, the check is also correct. If the problem is not solved correctly, students can find the source of the problem and will know what to correct or rethink.

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1.3 The Dienes-Varga method

1.3.1 Introduction

This part presents Zoltán Pál Dienes' theory of learning, his playful, experience-oriented learning method, and the use of games in the mathematics classroom. These methods and practices enhance practical application, playful, experience-based development and improve logical reasoning skills.

Zoltán Dienes and Tamás Varga's efforts were directed toward the promotion of playful mathematics teaching. Dienes' four basic principles (dynamics, constructivism, mathematical variability, perceptual variability) and stages of learning can be compared to Piaget's stages of conceptual development: free play, rule-based game, comparative structuring, representation, symbolization, and formalization. Dienes' method is a timely and gap-filling didactic tool. It playfully conveys and teaches mathematical concepts based on children's basic curiosity to explore the world. As a result, it provides a high-quality, interactive education process, meeting the needs of modern-day education. Its professional tools, world view, and pedagogy communicate such values as interdisciplinarity, experiential pedagogy, receptiveness, and experiential learning.

The aim of mathematics education is to make mathematics learning an engaging and exciting experience, which children enjoy and are motivated to engage in. The objective is to test and implement methods and tools which make maths lessons more efficient and appealing.

Another objective is to look for or create problems that cannot be found in textbooks, problems which are varied, ranging from easy to more challenging, from practical to more abstract, and which are adapted and made more tangible so that students find them more appealing and look forward to solving them. The objective is to find or create problems that children find captivating and which engage their energy, senses, and even limbs, thus stimulating not only their minds but their whole body. This process promotes a deeper level of assimilation of information.

The aim is to develop students' creative thinking and problem-solving skills; to teach children to apply their current knowledge to discover correlations; to encourage logical, critical and divergent thinking and reasoning. To examine open problems as well, which allow multiple questioning, as these problems can be approached from various angles.

An experiential learning environment should be provided, which gives the opportunity for discovery, enhances the desire for knowledge, intrinsic motivation, teamwork, getting to know each other, talent management and gaining experience.

The more methods are used, the more students can be reached. The more ways students are reached, the more rooted the subject matter will be (Kagan, 2004).

The aim is to use different active and interactive methods, various activities, but in particular those which bring mathematics closer to students, didactic games and some elements of the cooperative method.

The main point is to include students in the problem-solving process and help them be active and successful. Students should have a sense of accomplishment, they should not be afraid of trials and error, and they should heartily solve problems on their own. 'If we feed them only dry, abstract concepts, no wonder they will say: So what?... abstract thoughts, concepts can fairly quickly become insignificant...' (Finser, 2005, 40).

Students today are exposed to plenty of stimuli and they themselves can access a lot of information due to technology. As a result, it is essential to choose problems that motivate students. Problems have to be formulated in such a way that they relate to daily life and build on students' experiences. During the problem-solving process, students should learn how to highlight the essential information and what constitutes the core of the problem. Problem solving enhances students' logical reasoning skills as well as their ability to recognize and use patterns. Variety is essential when choosing problems as this helps students think 'outside the box'.

Unfortunately, the current system gives very little room for problems that improve skills, but not only for these. The current curriculum is overloaded. Teachers are forced to keep up with the topics on the curriculum, while students lose track from time to time. When this happens, all progress is in vain, as the foundation for further topics is missing.

Well-chosen problems develop children's skills in such a way that they will apply their knowledge in real-life situations more successfully. They will realize that the knowledge they acquire at school is not abstract and impractical and their ability to perceive problems also improves. The more personal, illustrated, and child-oriented the problem is, the more intrinsically motivated the children are.

1.3.2 Zoltán Dienes and the fundamental principles of learning mathematics

Zoltán Pál Dienes (1916–2014) was an internationally renowned mathematician and educator, sometimes also called a magician, one of the most outstanding figures of the 20th century pedagogy, who tried to do the impossible, namely to turn mathematics learning, which most children find something distant, into a playful, enjoyable and creative activity. '*My main aim was to deliver students around the world from the sufferings caused by maths lessons'* (Dienes, 2014).

Dienes devoted his career to the international development of mathematics teaching worldwide. He worked with hundreds of teachers and thousands of children on all continents, spreading understanding and arousing enthusiasm. He travelled almost all around the world, disseminating his methods on *games,* as an efficient tool in mathematics teaching.

His *fundamental principle* was that school activities should, by all means, build on children's play and activities, they should be infused with 'life', with concrete, real-life experiences.

According to Dienes, 'serious mathematics' learning can start much earlier than people would have thought. This can be accomplished through games, dancing and stories. There is a lot of mathematical experience in music, singing, rhythm, dancing, and singing-moving games (to name just a few: the spatial drawing of circle games, experiencing spatial directions, different symmetries, matching, forming sets in folk games, fractions in quarters, eights, beats, laying the foundation for the concept of number in increase-decrease games, etc...). This could be exploited in the musical aspects of the concept and it would be a meeting point between music and mathematics, even in the cultural context of Hungarian folk music.

Adhering to constructivist principles, Dienes claimed that children's activity is the crucial point in the learning process. The role of the teacher is that of instructor, facilitator and organizer. The source of knowledge is not the teacher but children's experiences, which are filtered through expectations generated by prior knowledge. This makes new information either integrable or conflicting in a productive way.

Dienes emphasized that external motivation (such as punishment and reward) should be replaced by internal motivation, and learning should become a joyful discovery for children (Dienes, 1966, 1973: 24–45). In order to achieve this goal, Dienes devised games which are adequate both from the point of view of mathematics and learning psychology (Bálint, 2015).

'These games –says Tamás Varga (1989:7) about the games devised by Dienes– (...) take intellectual effort, intensive thinking on the part of the players. They are the living refutation of the play – learning and play-work primitive contrast'.

Deep interaction with the environment enhances a more efficient coping with challenges (Fredrickson, 2001, 2009 in Bálint, 2015) and, of course, with productive conflicts. Dienes does not speak about flow. Nonetheless, there is a noteworthy analogy between the experiences of children playing the Dienes type games and those experiencing the phenomenon described by Csíkszentmihályi (Bálint, 2015).

Dienes points out that this method builds on the insight of the learner (Dienes, 1973: 44) without consciously bringing forth its occurrence, however. Dienes believes in 'miracles' and his observations confirm that these do happen gradually. His self-confidence reminds one of the maths teachers in a joke, who considered the miracle as an evident second step in solving the problem (Bálint, 2015).

Millions of children, from Australia to Canada, had the opportunity to discover the beauty of mathematics with the help of this playful method. Since children all around the world speak the same language (and this is what Dienes was interested in), he found his place everywhere. He put together the curriculum of elementary schools in several countries. The fundamental principle of his maths teaching was that using manipulatives, games, dancing and stories 'serious mathematics' learning can start much earlier than people previously thought. 'Give me a mathematical structure and I will turn it into a game' – he said. He firmly believed that children (under 12) are involved in constructive types of thinking, consequently he shifted the emphasis from analytical thinking – involving problems that most children could not understand- to playfulness. Besides the games he devised, he laid emphasis on music (listening and singing), stories, dancing and all kinds of movement. He devised numerous tools and games, but whenever he was around children, he adapted the problem to the physical and intellectual characteristics of the surrounding environment. Dienes professed that mathematics teaching and learning could not be a self-serving, abstract task; its primary objective should be the complex development of one's personality. He replaced individual and competing problem solving with cooperative games and group work. He was the first to openly state that there are differences between children in learning not only in terms of level but also of quality. He considered that children could access different mathematical concepts in significantly different ways. Consequently classifying students based on their abilities does not yield homogeneous groups (Dienes, 2014).

Dienes further developed the meaningful learning theory. He revealed that rote learning is less applicable in mathematics as in this subject, the emphasis is on the structure, not so much on the content. He considered that infinite reasoning is the essence of mathematical thinking. He was convinced that the transition to analytical thinking is made after 12 years of age, hence he believed that construction should come before analysis.

Zoltán Dienes formulated the fundamental principles of learning mathematics. He proposes the following principles in mathematics teaching and learning:

The dynamic principle. Preliminary, structured, practice and/or reflective types of games/activities must be provided. These serve as necessary experience from which mathematical concepts can eventually be built as long as each game is introduced at the appropriate time. We will see that this division can be further fine-tuned.

Though these games involve the use of concrete objects in the case of younger children, mental games can gradually be introduced to give them a taste of the most attractive game, which is mathematical research.

The constructivism principle. Games should always be structured to allow construction to precede analysis, which is usually absent from children's learning until the age of 12.

The mathematical variability principle. Concepts containing variables can be efficiently acquired if children experience many variations of variables.

The perceptual variability principle or multiple embodiment principle. 'In order to promote abstraction of the mathematical conceptualization children should be provided with many equivalents, yet perceptually different experiences' (Dienes, 2015, 47).

Dienes identified six stages of teaching and learning mathematical concepts: free play, games, searching for commonalities, representation, symbolization, and formalization. Together with his colleagues, he conducted experiments for several years and based on their findings, their theory can be fine-tuned considering the following:

Stage one. (Free play) Children and their environment mutually influence each other, consequently it is essential to carefully choose things that surround them. The environment should be as varied as possible. It should contain many mathematical relationships but also elements that enhance the development of other skills as well.

Stage two. (Rule-based game) Through their interaction with their environment, children realize that there are certain limitations. Certain rules govern events, some things are possible, and some others are not. This is how they discover regularities and realize that rules govern the world. Once children have noticed that rules govern events, they can invent new rules, apply them, modify or eliminate them. What makes a rule-based game exciting is that the goal can be achieved even when limitations are respected. Children's attention has to be drawn, in some way, to the fact that these rules are not intransgressible. In light of this, the creative child alters and develops the rules and receives more recognition than children who perform the task perfectly but stick firmly to the rules.

Stage three. (Looking for common properties, recognizing the common structure) When children have already played several games which contained the same structure, it can happen, though it is rather unlikely, that they become aware, on their own, of the common structure present in the games. The discovery of the common structure 'can be encouraged with the so-called dictionary technique. This method 'translates' one example of the concept into another example without altering the abstract properties common to all examples (Dienes, 2015, 49).

Stage four. (Illustration, representation) In order to consolidate the abstraction, children need help through representation. This will help them become aware of the commonalities present in all examples of the concept. Once they realized what multiplication is, they did operations which always resulted in the same number of objects. 'The dictionary technique helped them realize that they have used the same mathematical concept, namely multiplication, in all examples' (Dienes, 2015: 51).

Stage five. (Description, symbolization) Children become aware of the properties of multiplication. They describe their representation of the concept, which brings them a step closer to using an appropriate verbal or mathematical system of symbols. It is beneficial for each child to invent an individual system of symbols; however, they have to be familiarized with the mathematical system used in the textbook.

Stage six. (Formalization) This stage is meant to organize the descriptions generated in the previous stage. Some of the properties are considered fundamental. Rules are established for deriving other properties from the fundamental properties.

In most cases, schools do not follow the above-described procedure in teaching mathematics. It is very uncommon but possible that children are capable of abstraction even when they have minimal experience. It might happen that teachers and children have a similar way of thinking, which makes explanations more fruitful. Apart from these isolated cases, however, mathematics is not taught in the form of concepts building on each other, acquired through experiences, but rather in an associative way. This entails the use of different methods in different situations. A method is used in all situations which are similar to the one in which that particular method was used. However, if a slight change occurs, such as changing the letters, or a change in formulation, students assess the situation to be an unfamiliar one. This is due to the lack of transfer, i.e., applying the same example to different situations. Children cannot see the bigger picture, consequently, they do not solve the problem, or the results are incorrect. To implement the mathematics teaching presented in this part, the class has to be organized differently and the information has to be conveyed in a different way.

Teaching should be done individually or in small groups so that each student can grow independently, in a consistent way with their individual abilities and skills. In this type of education, the teacher can obviously not serve as the source of all information, since this is impossible in a class of thirty students. There have to be additional sources of information so that children can work on their own, find out what the next problem is, check their solutions and correct them, if necessary. This is where differentiated learning starts, when children are assigned tasks based on the stage of development they are in. This can be accomplished with the help of a layered curriculum. Children are given worksheets to work on. These worksheets have to contain problems that are connected to each other and lead to the formulation of a new concept, or the same concept can be presented through different means.

Concrete materials can make learning more efficient. Children follow instructions when dealing with course materials and achieve results through experiences. Problems have a step-by-step structure, which facilitates the understanding of the structure of the mathematical concept. When learning abstract notions, it is particularly important to make the best use of practical problems. These problems make concepts applicable.

In this context, teachers have to deflect from their traditional role. An authoritarian attitude is by no means beneficial, as children should feel free to ask questions. The teacher should take up the role of adviser. For example, if children do not understand a word problem, the teacher should reformulate it, or if children do not have the necessary knowledge to solve a problem, the teacher should choose a problem appropriate to their level.

The question might arise whether children would be willing to work independently in the absence of external factors and an authoritarian attitude on the part of the teacher. The answer lies in motivation. Another important question is what would create discipline and arouse interest in problem solving? The answer is not straightforward. Teachers cannot shift the responsibility onto the children. They have to communicate that they are there to help but also they are the ones responsible for the children. Using varied and engaging problems helps the teaching process. If children can sense the teacher's enthusiasm, they will have the same attitude and there will be no problems regarding discipline (Dienes, 2015).

1.3.3 Tamás Varga

In the 1960s Hungary, Tamás Varga and his team developed a new revolutionary method in teaching mathematics, a complex system of curriculum, teaching strategies, organisational methods and tools. He was actively involved, alongside Dienes, in the effort to renew elementary school mathematics teaching. They both asserted that a high level of abstract thinking could be attained through experience and solving real problems in a playful way. (Internationally this method is known as the Varga- C. Neményi method. In Finland, for example, the implementation of this method started in 2000). To their credit, Varga's team has realized that it is not enough to teach counting and measuring to elementary school children. They have to gain experiences in other areas of mathematics as well since there is a need for a prolonged experience-gaining period.

'According to Tamás Varga, children can be taught new topics as long as this is done playfully. We find many examples in his handbook for teachers, in further training materials and later in workbooks and textbooks, on how children's curiosity can be aroused for such areas of mathematics as combinatorics, sets, mathematical logic, functions, etc. Varga also recommended tools for teaching children, such as the Babilon construction game for developing spatial cognition, or the Dienes set for teaching the system of numbers' (Oláhné, 2018, 20).

The complex mathematics education experiment stood out from other educational experiments due to its comprehensive and coherent content and methodological design. It served as a basis for the new curriculum introduced in 1978 in several stages to higher levels.

The fundamental principles of the Tamás Varga method can be summarized as follows:

- Personalized, age-appropriate skills development
- Strengthening mathematical thinking by application (in other school subjects as well)
- Replacing external motivation with internal motivation
- Active learning experiences by involving versatile perceptions
- Inductive learning
- Spiral curriculum
- Replacing the narrow-minded arithmetics-geometry teaching by adding different areas of mathematics to the curriculum.
- Methodological diversity, using tools and games

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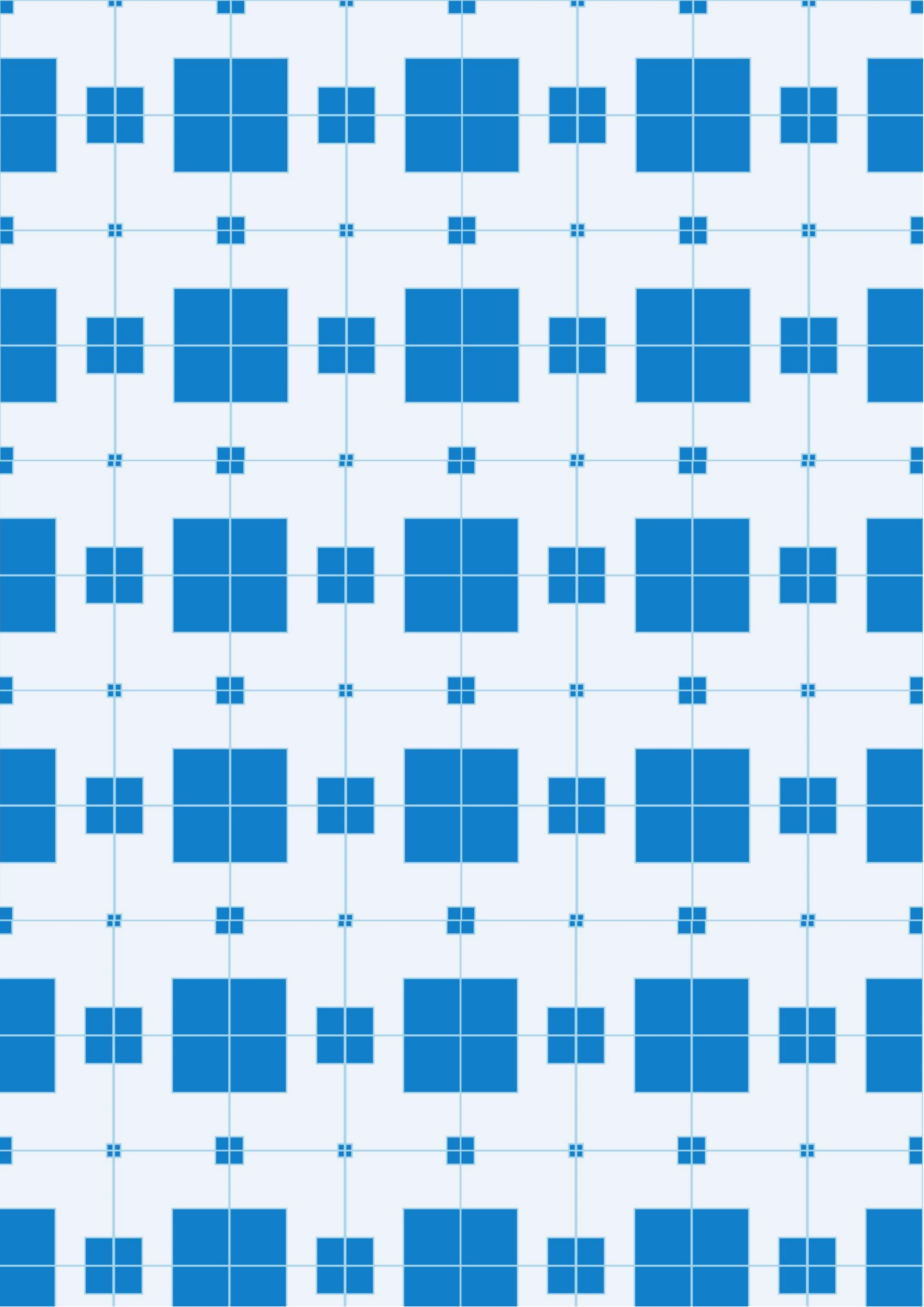

1.4 Inquiry-based strategies in mathematics education

Problem-Based Learning (PBL), Inquiry-Based Learning (IBL), and Project-Based Learning (PjBL) are all student-centered teaching paradigms that promote active learning and critical thinking mainly through investigation. Many authors say that all three types are effective learning models. Inquiry-based learning revolves around a central question that covers the curriculum outcomes and emphasizes the process of discovery. Students are challenged to address authentic, possibly intransparent, open, real-world problems and build answers through problem-based learning. The eventual goal of project-based learning is to create a valuable, tangible result during a long-time program. All three strategies present students with exciting problems to think about. However, these categories are not without overlap, nor are they used uniformly in the literature.

1.4.1 Inquiry-based learning

Inquiry-based learning has its roots in the constructivist American pedagogical movements of the 1960s. Historically, it is related to research-based and design-based learning and problem-based learning, the project method, and complex instruction. All these approaches are present in the current practice of inquiry-based learning: inquiry-based learning can be implemented through projects, but also through research-based or design-based learning. The essence of inquiry-based learning is that it seeks to 'put learners in a situation': that is, to create situations in which learners are active participants in the learning process, solving (preferably authentic real-life) problems, exploring the problem, gathering information, investigating, evaluating alternatives, designing experiments, modeling, reasoning and discussing with other students (Réti, 2011).

Although there are many forms of inquiry-based learning and many varieties from the perspective of mediated science, most of them highlight four aspects of the learning process:

1. problem-focused activities – where the aim is often not to find a single correct answer but to explore a complex system of issues or phenomena;
2. inquiry, experimentation, information-gathering activities;
3. self-regulated learning cycles, promoting learner autonomy;
4. argumentation, discussion, communication, and presentation and communication of results.

The above aspects can be implemented in a wide variety of ways, both in individual student work and in group activities. Inquiry-based science learning can also be linked to the pedagogy of sustainability: in open discovery, the student deals with a problem relevant to him/her; the teacher acts as a facilitator to support the learning process (Réti, 2011).

1.4.2 Inquiry-based learning in mathematics

Mathematics is a highly formalized, deductive science. For thousands of years, the mathematical knowledge of humanity has been a similar form: a set of definitions, theorems, proofs, and propositions. Since Euclid, the emphasis in mathematics has been on deductive reasoning and its role in justifying mathematical knowledge; this is one of the most outstanding mathematics achievements. Consequently, the emphasis on theorems and proofs, and proof in general, has helped promote traditional absolutist views of mathematics education. However, the nature of mathematical thinking is not structured in that way; the structure of definitions, theorems, and proofs is not the structure of mathematical thinking. Another trend in the history of mathematics education emphasizes *discovery and problem-solving*. The followers of this trend emphasize that a large part of mathematics is human problem solving, which has implications for education. These include:

- Inquiry and investigation should be integral to the classroom mathematics curriculum.
- The pedagogy employed should be process and inquiry-based (Ernest, 1991).

Inquiry-based mathematics education (IBME) refers to a student-centered paradigm of teaching mathematics. IBME encourages pupils to work like mathematicians and scientists. Their responsibilities include observing and analyzing data and interpreting, assessing, expressing, and arguing their solutions. The teacher's role differs from conventional teaching methods: explanations by teachers, illustrations, and exercises no longer dominate pedagogy. Instead, students work together on interconnected and challenging assignments. Teacher responsibilities include using students' previous knowledge, pushing them with probing questions, facilitating small group and whole-class discussions, promoting opposing views, and helping students link their ideas (Dorier & Maass, 2014). In such a setting, the teacher's role differs from traditional teaching approaches: pedagogies shift away from a 'transmission' orientation, in which teacher explanations, illustrative examples, and exercises predominate, toward a more collaborative orientation, in which students work together on 'interconnected,' 'challenging' tasks. The teacher's role in this situation includes making constructive use of students' prior knowledge, challenging students with effective, probing questions, managing small group and whole-class discussions, encouraging the discussion of alternative viewpoints, and assisting students in making connections between their ideas.

There are various inquiry-based strategies in mathematics education: guided discovery, problem-solving, and the investigatory approach.

- In guided discovery the teacher poses problems or chooses a situation with a goal in mind. The teacher also guides students toward the solution. Students follow the guidance.
- In the problem-solving approach, the teacher poses problems but leaves the method of solution open. As a result, students find their way to solve problems.
- In the investigatory approach, the teacher chooses the starting situations or approves students' choices. Then, students define their problems within the situation and attempt to solve them in their way (Ernest, 1991).

1.4.3 Problem-oriented approach and the Poly-Universe

When using the Poly-Universe tool, the principles of IBL can be applied, usually in the following modified form, which we call the problem-oriented approach (Kovács & Kónya, 2019). Three properties characterize the problem-oriented approach to learning mathematics:

- students analyze a mathematical problem situation;
- students critically adapt to their own and their classmates' thinking;
- students learn to explain and justify their thinking (Csíkos, 2010).

The teacher should look for appropriate questions that correspond to the curriculum and are suitable for group work and discussion. There are also attempts to set authentic problems taken from real life (Hoffmann, 2020).

Here is an example for seventh graders. When asked about the properties of the square element (Figure 1), students may notice that the corresponding vertices of the three interior squares fit on a straight line. Otherwise, it is helpful to have the element drawn with the pupils on square grid paper or on a geoboard to help them recognize the property. Explain why this is so! The Think-Pair-Share cooperative method is very suitable to organize the discovery of this problem. The justification requires the recognition that all three vertices lie on one of the diagonals of the square.

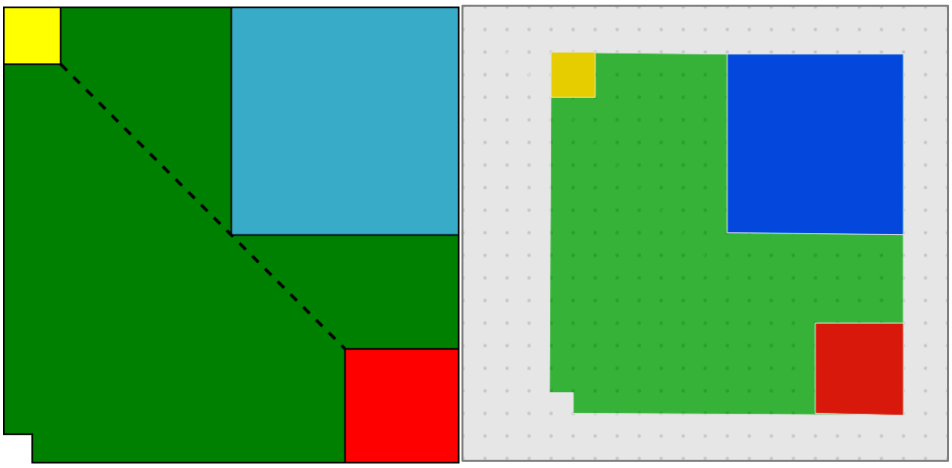

Figure 1: The square element and its drawing on Mathigon's grid

In the original version of the Poly-Universe, the ratio of the sides of the corresponding elements is 2. We can experiment with the ratio using dynamic geometry tools and pose different problems. In grades 9–10, it can be an interesting problem to look at the ratio at which the elements will meet. It is worthwhile to use GeoGebra to construct the elements and then use the slider to find the right positions, either before or after the calculation (Figure 2). The top right figure shows a second-degree problem, and the result relies on the golden ratio. The other problems are third-degree, so we need some technological application to solve them. The bottom right figure, reminiscent of Viviani's theorem, offers an exciting discovery also.

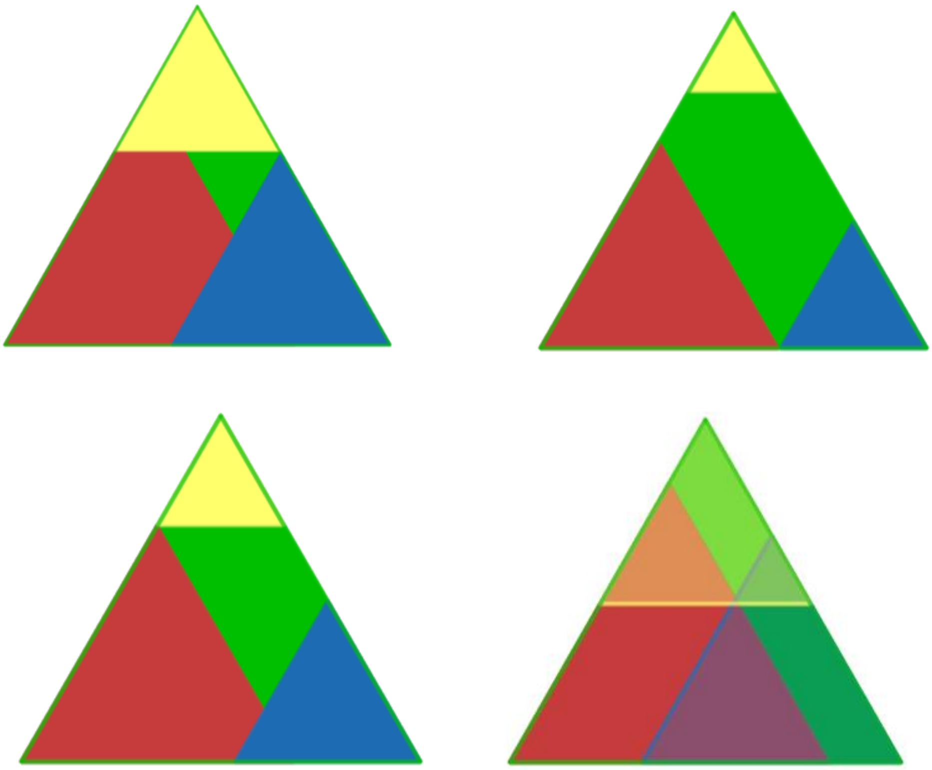

Figure 2: Some triangle pieces corresponding to different aspect-ratios

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1.5 STEAM Education

There is a growing emphasis on the role of multiple creativities in today's society. Companies in the labour market place a higher value on their workers' flexibility, fast and smart decision-making abilities, creative and critical thinking skills, innovation, teamwork, communication, and entrepreneurial spirit. To respond to these growing expectations, students must learn new ways to approach problems, gain skills and competences, and create and use tools in innovative ways.

Flexibility and variation ought to be the goal of education – preparing young people to apply tools creatively, in virtual and real community with others, across former disciplinary boundaries. It should not only be possible but imperative for children to attend school to improve the world. Authentic projects based on real-world and real-life problem-solving provide opportunities for learners to be appraised based on their contribution to the success of their communities in terms of solving social problems or contributing to sustainability.

STEAM is an acronym for Science, Technology, Engineering, Arts, and Mathematics. While there is a growing body of literature focusing on various aspects of STEAM education (Belbase et. al, 2021), ‘the term STEAM is as contested in its understanding as it is diverse in its practice’ (Burnard & Gray, 2021). Despite the diversity of understanding and practices, it can be said that STEAM refers to an educational approach promoting integrated teaching and learning by studying phenomena and topics from multiple perspectives.

The primary purpose of combining various perspectives is to move beyond the traditional subject-based education and achieve cross-, inter-, multi-, or transdisciplinary connections while building up the learning experience. Arts and artistic processes contribute directly to the inquiry by creating, performing, and connecting scientific content and methods with arts-based content and pedagogies.

Implementing STEAM Education engages and motivates students through relevant, meaningful, playful, and multisensory learning experiences. These experiences emerge from the individual and collaborative design and problem-solving activities (Figure 1). The STEAM activities are characterized by higher-level thinking, process-over-product perspective, skills and competence-development over memorizing facts, hands-on activities, embodied learning over solving textbook problems, and cultural and emotional literacy development. STEAM education is usually organized in project-based formats and encourages divergent (‘out-of-the-box’) thinking and authentic assessment. (Cofield, 2017) Breaking down ‘subject silos’ by developing the multidisciplinary and phenomenon-based forms of learning – where the Arts are integrated into problem-solving – adds a creative and human dimension which can bring learning to life.

According to So et al. (2019), teachers' educational competency in STEAM pedagogies is reflected in knowledge, skills, attitudes and competency in creative convergence. As a background knowledge, it is recommended that STEAM teachers are understanding education policy, have an overview of integrative knowledge and integrating technology. On the skills level, it is recommended that teachers are prepared for enacting STEM/STEAM classes. This includes establishing cooperative/collaborative learning, providing problem-based and inquiry-based learning, supporting individualized learning and ready for leading the creative and authentic assessment/reflection sessions.

In terms of attitudes, it is recommended that teachers have a positive attitude and recognize the need for STEM/STEAM education. It is required to appreciate art, have a positive attitude towards science and accept novel technologies. They also need to practice coming up with new ideas by seeing and combining the relevance of existing knowledge sources and applying such transdisciplinary knowledge to real-world problems.

Figure 1: Finnish teachers, Merja Sinnemäki (left) and Leena Kuorikoski (right) introducing the results of their students' Polyuniverse STEAM project, merging Dirk Huylebrouck's fractal tree design with Saxon's Poly-Universe modules. Photo: Kristóf Fenyvesi

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1.6 Games in learning

People have always played games and as Bright, Harvey and Wheeler (1985) stress out: '…games have evolved along with civilization. It seems that games survive and are played because people enjoy playing them'.

Game is a source of joy, it is interesting and activating. It provides the players with a myriad of opportunities for exploration, brainstorming, constructing, problem-solving, communicating and collaborating with other people. Playing games the participants become flexible, strengthen relationships among each other and they learn to resolve conflicts.

Games can be an excellent tool for building the students' interest, especially, when students do not like the school subject they are learning. By using games, teachers can urge students to deal with problems and as the games are challenging and entertaining, they always motivate students to learn.

Oldfield (1991) gives the following definition of Mathematical Game:

'1. It is an activity involving: EITHER a challenge against a task or one or more opponents
OR a common task to be tackled either individually or (more often) in relation with others.
2. The activity is governed by a set of rules, and has a clear underlying structure to it.
3. The activity has specific mathematical cognitive objectives.' (p.41)

If we replace the word mathematics in this definition with the name of any school subject, we will get the definition of a game for that school subject.
It can be concluded from the above mentioned that the learning, in which games are used, fully satisfies the criteria for learning based on constructivist learning theory.

Davies (1995) summarized the advantages of implementing games in mathematics:

- 'Meaningful situations – for the application of mathematical skills are created by games;
- Motivation – children freely choose to participate and enjoy playing;
- Positive attitude – games provide opportunities for building self-concept and developing positive attitudes towards mathematics, through reducing the fear of failure and error;
- Increased learning – in comparison to more formal activities, greater learning can occur through games due to the increased interaction between children, opportunities to test intuitive ideas and problem solving strategies;
- Different levels – games can allow children to operate at different levels of thinking and to learn from each other. In a group of children playing a game, one child might be encountering a concept for the first time, another may be developing his/her under-standing of the concept, the third one could be integrating previously learned concepts;
- Assessment – children's thinking often becomes apparent through the actions and decisions they make during a game, so the teacher has the opportunity to carry out diagnosis and assessment of learning in a non-threatening situation;
- Home and school – games provide 'hands-on' interactive tasks for both school and home;
- Independence – children can work independently of the teacher. The rules of the game and the children's motivation usually keep them on task.'

For creating Successful Classroom Games the hints which were formulated by Aldridge and Badham (1993) can be very useful:

- 'Make sure the game matches the mathematical objective;
- Use games for specific purposes, not just time-fillers;
- Keep the number of players from two to four, so that turns come around quickly;
- The game should have enough of an element of chance so that it allows weaker students to feel that they have a chance of winning;
- Keep the game completion time short;
- Use five or six 'basic' game structures so the children become familiar with the rules – vary the mathematics rather than the rules;
- Send an established game home with a child for homework;
- Invite children to create their own board games or variations of known games.'

The mentioned advantages and hints can be useful guides for all teachers and educators (not just math's teachers).

Poly-Universe sets have a big potential for creating a variety of games for learning notions of different fields and for students' emotional and social development.

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1.7 Visuospatial Skills

Spatial intelligence has been a major area of research since the 1930s. Though there are numerous approaches to visuospatial abilities, no universal definition has been formulated. The terms *spatial abilities, visuospatial abilities*, *spatial cognition*, *spatial intelligence*, and *spatial knowledge* are used interchangeably. The same applies to the usage of the terms *skills* and *abilities* (Babály, 2020).

The most often quoted definitions are:

1. '*the ability to mentally rotate, manipulate, and twist two- and three-dimensional stimulus objects*' (McGee, 1979)
2. '*performing tasks that require mental rotation of objects, the ability to understand how objects appear at different angles, and the ability to understand how objects relate to each other in space*' (Sutton & Williams, 2007)
3. '*visuospatial abilities facilitate wayfinding, imagining objects rotated at different angles, and remembering object's position in the environment*' (Lawton & Hatcher, 2005)
4. One of the most often quoted definitions is: '*the perception of visual shapes and positions of objects, forming mental representations, and the manipulation of these representations*' (Carroll, 1993).
5. '*Visuospatial ability is the ability to perceive two- and three-dimensional shapes, to understand the relations between the object perceived, as well as to solve problems*' (Séra, Kárpáti, & Gulyás, 2002: 9).

Until the 1960s, spatial cognition involved two factors: perception and visualization. Subsequent research focused on identifying various components of spatial ability, based on distinct spatial operations. First it was the ability to rotate mental representations which was isolated from visualization (Vandenberg & Kuse, 1978). At present there are five spatial components accepted by most researchers:

1. **Spatial perception**: also known as spatial recognition, encompasses the ability to receive and interpret visual stimuli.
2. **Spatial visualization**: this label is typically attached to complex spatial tasks consisting of multi-step manipulations, or to the ones that cannot be classified under any other components. The ability of choosing optimal strategies and the ability to quickly switch between these plays a key role in solving complex problems. (Mental transformations and mental rotations can be included here.)
3. **Mental rotations**: an imaginary movement of two- and three-dimensional shapes, where the whole object is transformed by rotating. (In the case of mental transformation only a specific part of the object undergoes some kind of transformation.)
4. **Spatial orientation**: this component is well-defined in the literature of the field. As a rule, it is defined in relation to mental rotation. In the case of mental rotation the object is mentally moved, whereas with spatial orientation a given viewpoint is moved mentally while the object remains fixed in space. It presupposes skills which enable thinking about the world in spatial terms. We can perceive spatial layouts and can follow the changes that occur (e.g. determining direction and distance). Séra, Kárpáti & Gulyás (2002) points out the important role of the observer's frame of reference: (1) egocentric (location of object is specified with respect to our position/viewpoint), (2) allocentric (we assume a viewpoint independent of our position).
5. **Spatial relations**: is the ability to rapidly and correctly rotate or create a mirror-image of the mental representations of an object. The definition itself suggests a significant overlap with the mental rotation component. Spatial relations is identified as a subskill in models which do not distinguish mental rotation as an independent factor (thus, we speak of renaming) or it is presented as a subcomponent of spatial relations (Babály, 2020).

Nagy (1998), in his study on the system and development of cognitive abilities, distinguishes three forms of learning: (1) exploration, (2) trial and error, (3) play. *Exploration* gives way to *knowledge acquisition,* which is identified as an independent skill, and it *encompasses looking for, selecting and retrieving information.* Considering the large proportion of information retrieved visually, understanding visual language is essential not only to the arts or other related fields (e.g. design, architecture) but to all disciplines. The author emphasizes visual communication as part of cognitive communication and identifies graph skills. He defines component skills (closely related to spatial cognition) such as: size perception, depth perception, structure perception, dynamics perception, as well as the ability to render these. *Trial and error* plays an important role in discovering new information and in *problem-solving skills*. The significance of *play* (especially creative and construction play) is pointed out in relation to *creativity* development. It is emphasized that the creative process results in a new product and new knowledge. On the issue of creativity being operated only through other cognitive abilities (problem solving, learning, and reasoning skills) the author states the following: 'I did not use to consider creativity as belonging to cognitive competences, however, I am more and more inclined to classify it as such' (Nagy, 1998: 12).

1.7.1 The development of visuospatial abilities

Piaget (1970) distinguishes three stages of development concerning spatial cognition:

1. In the first stage we learn to differentiate between objects based on their shape and distance (isolation, recognizing groups). At the age of 3–5 children develop the ability to integrate information about spatial categories that are formed along two dimensions.
2. In the second stage three-dimensional objects are imagined from differing viewpoints, movements and spatial transformations are perceived (grasping unfamiliar, complex shapes might pose a problem at this stage).
3. In the third stage spatial relations, sizes and distances are visualized, as well as tasks are carried out involving these internal representations (e.g. rotation, mirror-image formation, and joining).

In the early stages of development progress is determined by movement, action, tactile and other sensory experiences. Later, symbolic cognition, the use of language, gestures, maps and patterns will become increasingly important. Spatial representations allow for more information to be gained than direct experience.

The tasks of the second and third stage, i.e., spatial perception, constructing mental images and tasks carried out with internal representations might pose problems even in adulthood. The reason for this is that solving spatial problems is a multi-step complex process requiring the systematization of different cognitive operations (e.g. comparison, visual memory, integrating viewpoints, etc.) (Tóth, 2013).

Though the education system puts great emphasis on developing maths skills, the proportion of geometry teaching materials, which would serve the development of spatial cognition, is continuously decreasing. The mathematical literacy test of the OECD PISA survey (2012, 2013) contains a number of items which can be solved, partially or entirely, by using spatial abilities. There are eight different types of problems in the '*space and shape*' category which test spatial skills:

- estimating the area of a floor plan
- mental rotation of a building
- calculating roof area (imagining a 3D rendering based on 2D projection images)
- perception of spatial positions, relations
- estimating area and distance
- changing between mental viewpoints
- mental transformation (mental paper folding) (Babály, 2020)

Investigating visuospatial abilities is an active research area focusing at present on two age groups. The main aspects of the development of spatial cognition are examined in the age groups of preschoolers and elementary school students, while studies targeting young adults in higher education focus on identifying deficiencies in spatial abilities, as well as on developing these skills.

International research results show that spatial cognition can be successfully developed until the age of 18 and among young adults enrolled in higher education. The assessment of various programmes typically involves pre- and post-testing. Performance enhancement rates are determined by the duration of the development programme and the frequency of activities. They are influenced to a lesser extent by the types of problems used and the learning environment. It poses a problem that longitudinal studies focusing on the development of visuospatial skills are rather scarce, and so are results of data collection using qualitative research methods (Babály, 2020).

It might be important to point out that as regards the activity types enhancing the development of visuospatial abilities, non-formal and informal learning are as important as formal education. The activity types presented below can make the greatest contribution, in an indirect way, to the development of spatial abilities.

1.7.2 Construction play

The importance of developing construction skills is emphasized even at preschool age as experiences gained in the preschool years have a long-lasting and wide-ranging impact on children's ensuing development. When discussing the educational role of building games it is inevitable to mention Froebel's pedagogical activity. Besides designing educational play materials known as 'gifts', he also attached strict instructions and methodological descriptions to them.

'In Froebel's children's garden learning does not take place through speaking- and cognitive exercises but through movement, activity and creativity' (Szabolcs & Réthy, 1999: 364). Most of the toys were monochromatic, which shows that emphasis was placed on the structural relationships between geometric shapes, stimulating analytical interpretative spatial thinking, while the Maria Montessori method uses spontaneous activity for sensorial learning. Visual experiences of materials, shape, and color, thermal, tactile, baric and the auditory senses play a crucial role.

Construction games experienced a revival in the 1950s as scientists realized that they can serve the development of mathematical, physical, and chemical literacy and thinking. The United States government called for the introduction of these types of games in school education in order to develop children's knowledge in mathematics and natural science. Building games can be well integrated into current educational trends as gaining experience through play and active learning methods are encouraged. McKnight and Mulligan (2012) consider that the role of building games is to reflect on children's intuitive and informal learning. Furthermore, they can also be used to identify students' way of thinking and their skills. (In their research they used open-ended problems to assess children's problem solving skills.)

When a new game is introduced in the teaching-learning process, the question arises how to create a balance between the developmental effect of the game and experiencing a sense of enjoyment. Children do not necessarily enjoy playing games which best serve the educational purpose. There is growing interest amongst researchers when it comes to game-based learning. They are trying to explore methods which allow for the integration of different types of games into school education (Babály, 2020).

Today the toy industry offers a large variety of games meant to develop spatial ability, as well as a vast number of logic games. We can distinguish three types of traditional building games: (1) symbols, letters, words on building blocks; (2) simple, abstract (geometric) shapes; (3) models, architectural structures. These three types have different, sometimes overlapping, functions. The games belonging to the first group can strengthen the synchronization of conceptual and visual thinking, while architectural models can help refine the notions of proportion and scale. Logic games developing mental spatial abilities (e.g. mental rotation, transformation) constitute a separate category, which is currently experiencing a revival. These can be implemented into the teaching-learning process even with older children. However, in Hungary games developing skills are used mostly by special pedagogues, speech therapists and less often by kindergarten teachers, elementary school teachers and language teachers (Babály, 2020).

1.7.3 Crafts and DIY

Developing spatial skills does not necessarily require expensive tools. Simple games can be just as effective and enjoyable. For instance, origami can be used to develop such a complex spatial task as mental transformation (Pataky, 2012). However, modelling and building problems, which play an important role in refining mental representations, are often left out of the curriculum due to lack of resources or time.

1.7.4 Computer games

Videogames can enhance the development of spatial abilities, however their efficiency depends to a great extent on their type, frequency and length of usage. Quaiser-Pohl, Geiser and Lehmann's study (2006) reveals that action and simulation games also have a positive effect on rotation tasks. However, this correlation was verified only with the male sample, who had devoted more time to playing these games.

In the research, for developing and measuring partial skills such as spatial orientation and mental rotation, tasks in a virtual space, which use visually displayed computer games, are designed. (Andersen et al., 2012). Computer games' positive effects expand on the development of basic skills, which have strong connections with spatial abilities, such as spatial perception, attention (ex. resolution, field of vision), selective attention, short-term visual memory, object tracking (Spence & Feng, 2010; Castell et al., 2015).

The report on the OECD/CERI Project entitled '*New Millennium Learners'* (2008), which investigated the effect of digital technologies, highlights the role videogames play in the development of cognitive abilities, more specifically in the development of visuospatial abilities. It is pointed out that digital media and computer games contain features that provide an opportunity for enhancing memory and attention, interpreting visual information, as well as constructing mental representations. There is a lack of research on the transfer effect of the sub skills enhanced by video games.

1.7.5 Sports

Athletes presumably have spatial experience which leads to the development of above-average abilities. The question is, however, which sports develop which spatial abilities.

We find examples for research, which show that the athletes outperform the results of the non-athletes, for example in the examination of Habacha, Molinaro és Dosseville (2014), which contains mental rotation exercises (277 people, 18–31-year-olds). Athletes probably have a lot of visual experience, which lead to the formation of above-average competencies, however more studies are required to determine what kind of spatial skills are developed by different sports.

Since not much research has been done on comparing different sports, and existent research has been carried out on a small sample, there is still a lack of reliable research evidence on which types of activities (e.g. dance requiring focused coordination, gymnastics developing a sense of balance, handball or hockey requiring a quick assessment of spatial relationships) are more efficient in enhancing spatial abilities.

1.7.6 Arts

Research results suggest that art activities can have a positive effect on the development of visuospatial and other cognitive skills (Winner, Goldstein, & Vincent-Lancrin, 2013). Convincing evidence is provided mainly by studies on musical education (Gombás, 2014). There are a number of researches attesting a causal relationship between learning music, visuospatial abilities and achievements in mathematics (Hetland & Winner, 2004; Catterall, 2005). The goal of the Agam Programme was to develop preschool children's visual thinking. (Razel & Eylon, 1990) The two-year long development programme, containing various tasks aimed at teaching the basics of visual language, had a positive effect on the cognitive skills of 4- to 5-year-old children. It strongly strengthened the numeracy and writing skills of children starting elementary school (Babály, 2020).

The education system considers, even today, that visual thinking is a secondary intellectual activity. Drawing and construction is deemed childish or 'primitive'. There is still a lacuna of research in the area of three-dimensional spatial representation. The most frequently found terms in pedagogical documentation are building, constructing, planning and designing. The literature on spatial abilities uses the term 'construct' in a rather restrictive interpretation. On the other hand, the literature on visual abilities uses terms denoting a wide range of skills (e.g. construction includes the use of materials and tools, and creativity or expressivity). Since studies focus on complex creative processes, investigations target only certain aspects of spatial representation (typically structural vision, and relationships between forms).

In Hungary, on a national level, four large researches have been carried out recently on the development of the construction-design ability and on the identification of typical levels of knowledge at different age groups:

1. Leonardo Programme: the development of visual creativity and processing skills between the ages of 7 and 12;
2. the structure and development of the construction-design ability between the ages of 12 and 16;
3. building and designing objects between the ages of 6 and 12;
4. the development of artistic ability between the ages of 3 and 7. The results of the Leonardo Programme are of outstanding importance. Both phases of the research focused strongly on questions related to the development of spatial abilities (between 1991–1993 and 1993–1995). They assessed the impact of five educational programmes, which contained drawing tasks but also construction-design tasks. These provided an opportunity to compare the developmental effect of two- and three-dimensional spatial representation. The author concluded that *the most efficient way to develop visuospatial abilities is using three-dimensional modelling problems and real spatial experiences.*

The four Hungarian studies yielded a description of the subskills of the construction-design skill, which were identified with measuring tools well applicable in school education as well. The studies provide an insight into the peculiarities of developing two- and three-dimensional spatial representation. Furthermore, they advocate efficient development by contributing collections of problems (Babály, 2020).

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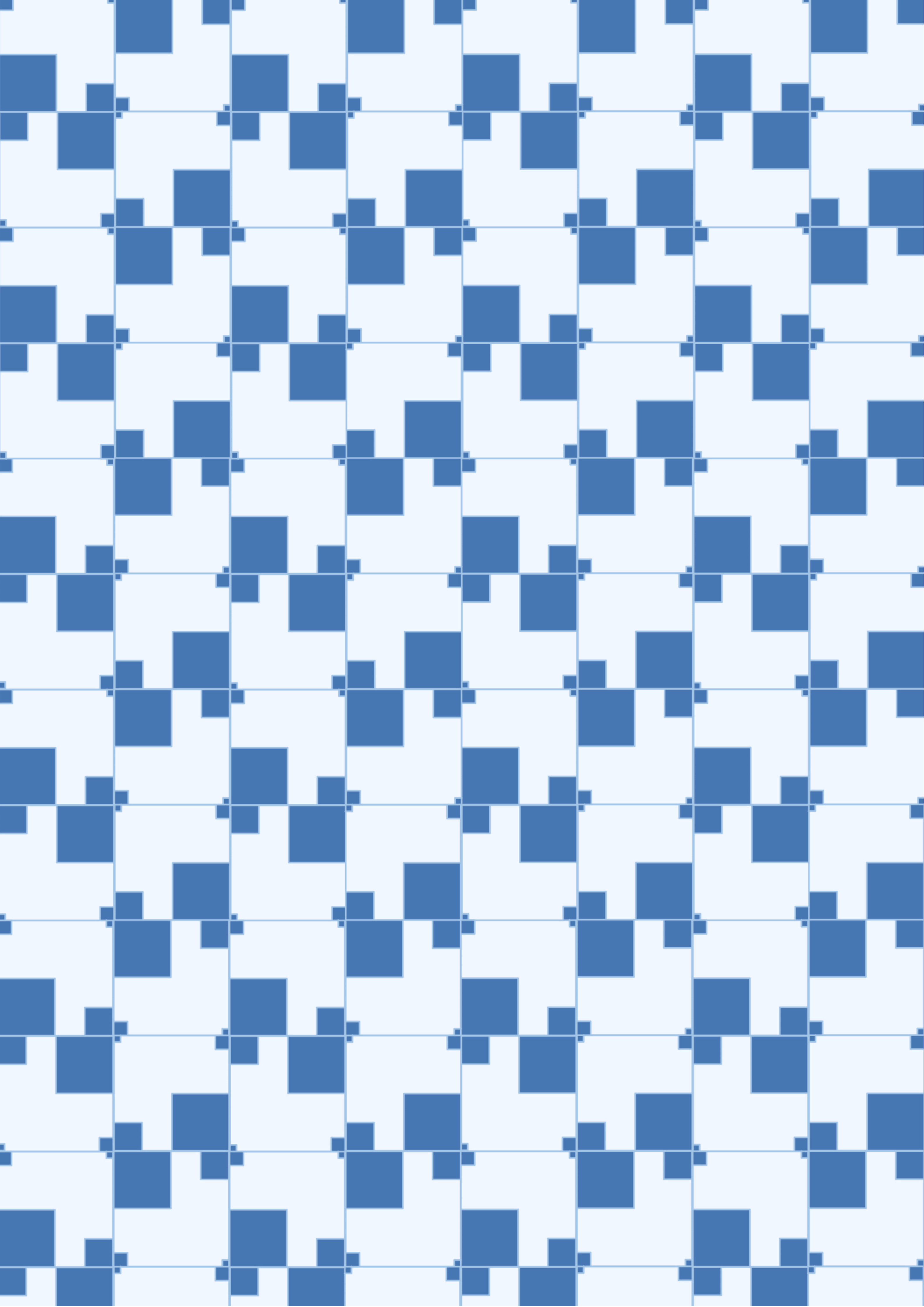

1.8 Motivation and engagement on learning

Good teaching includes students how to learn, how to remember, how to think, and how to motivate themselves (Weinstein & Meyer, 1986, p.315).

Introduction

Many students express beliefs and expectations of controlling outcomes that identify with debilitating or low motivation patterns for success, namely in mathematics, saying that mathematics is difficult, and anxiety around learning mathematics is frequent. Often, they attribute failure to lack of capacity or difficulty in the subject, generalizing an impotence response to subjects that they would even be able to learn, resulting in situations of discouragement or helplessness. Indeed, we can distinguish a set of cognitive-motivational variables, which include causal attributions, outcomes control expectancy and goal orientation, which are particularly pertinent in the context of learning mathematics (e.g. Schunk & DiBenedetto, 2016).

It should be noted that the investigation about the motivational processes in education is nowadays one of the more relevant topics of study. Under the influence of socio-cultural perspectives, we witness an evident change of focus in the study of motivation and learning, from the individualist approaches, which conceive the motivational processes as personality traits *(trait specific)*, to the contextualized approaches, that enhance the adaptive nature of students' motivation and learning *(task specific)* (Cordeiro, Lens & Bidarra, 2009). Indeed, much of the research that developed from the 1970s onwards (70s), under the aegis of cognitive and behavioral paradigms, remained prisoner of a trend towards the study of school motivation centered on the processes and characteristics of students, on characteristics of learning, behavior and teaching contexts, instead of studying them in interaction, to understand the way they influence each other (Perry, Turner, & Meyer, 2006).

However, the cognitive paradigm gave origin to many theories and models of motivation, such as the Expectancy x Instrumentality x Value – VIE theories, Achievement Goal Theory AGT and Self-determination Theory – SDT, generating a huge investigation about the motivation of achievement in the classroom that reflect a general change in the investigation about teaching, in a way to create more situated models or ecological motivation and learning (Perry, Turner, & Meyer, 2006). It appears, for example, that cooperative learning structures favor more adaptive motivational patterns, in the sense of fostering attributions to effort and the perception of self-efficacy, and an orientation towards the task, rather than towards the result or performance, contrasting with competitive structures, emphasizing the influence of contexts on the motivation for learning. The relationship between motivation and academic performance has been studied, with higher levels of income being associated with higher levels of motivation, with motivation often considered as a mediating variable, but which can also be considered an end or objective to be achieved in educational terms. The importance of non-academic measures, soft skills, socio-emotional or transversal skills has been highlighted, with emphasis on the operationalization of constructs such as self-regulation, persistence, in addition to engagement and emotional well-being, among others, which are associated with academic success (Moore, Lippman & Ryberg, 2015), and which have been identified as 21st century competencies by OCDE 2030 framework , included in the Students' Profile by the End of Compulsory Education, in Portugal.

In this text, we intend to address the role of motivational cognitions, that is, subjective beliefs and perceptions, within the framework of cognitive theories of motivation, which differ from those that focus on a quantitative and mechanistic perspective, depending on stimuli or reinforcements, or on internal or stable characteristics, the behavior or approximation to certain activities, designated by incentive theories and needs theories, respectively (Lemos, 2006). It is intended, therefore, to account for the qualitative aspects of motivation that focus on the processes that mediate external and internal determinants and behavior, referring to systems of cognition, which involve the way in which students, as well as teachers, approach tasks, they process information about the situation, and interpret their achievement or performance.

It is assumed that the same variables influence the motivation and behavior of both students and teachers in achievement contexts. It is also assumed that contexts, namely the organization and dynamics of the classroom, act on motivation by evoking certain cognitions and emotions, such as expectations of success, self-efficacy and satisfaction, which influence motivated behavior. In this sense, we will highlight the goal orientation theory, the causal attributions theories and expectancies for control of reinforcement, and we will analyze the conceptual distinction and operationalization of the concept of engagement, associated with motivation, a multidimensional concept, which completes this approach.

1.8.1 Goal orientation and motivational cognitions

Achievement Goal Theory: Main concepts

The focus of analysis to the Achievement Goal Theory is due to core aspects, namely because it is assumed as one of the theories (along with the Expectancy-Value Theory) that have generated a wider range of investigations about the achievement motivation in classrooms, and because it considers explicitly the influence of teachers and educational contexts in student goal adoption, contributing to the study of content or quality of motivation and the understanding of motivation processes and learning and students' academic success.

Several researchers of motivation psychology, development psychology, and psychology of education have developed and analyzed constructs associated to goal orientation with the aim of studying students' learning and development in academic tasks and in classroom situations (cf. Cordeiro, Lens & Bidarra, 2009). According to the Goal Orientation Theory, students may pursue learning goals (learning or mastery goals) or performance goals, which are sometimes called something else. However, they refer mainly to learning orientation, which is valued as an end in itself, in contrast to an orientation towards demonstration of ability, which is associated with different attributions and perceptions about self-efficacy, which translate into adaptive or debilitating motivational patterns (Lemos, 2006).

In fact, while in task orientation the learning outcome is associated with the effort made, thus increasing self-efficacy, in result or performance orientation there is a concern with demonstrating the ability to face oneself and others, which increases anxiety, the avoidance of challenging tasks, and may lead to learned helplessness, when failure is attributed to lack of ability. These two types of goals are also associated with different emotions and different attitudes and strategies towards learning (Cordeiro, Lens & Bidarra, 2009; Lemos, 2005). Research on goal orientation theory also points to the existence of a consistent pattern of results towards the positive consequences of a learning orientation in school contexts, which is manifested in effort, persistence, use of metacognitive strategies and a deep approach, with greater involvement in self-regulated learning (Cordeiro, Lens & Bidarra, 2009). However, some authors point out that achievement or performance goals may not be maladaptive, as long as they are not performance avoidance goals, and may take the form of approach or involvement (performance approach), in which the aim is to demonstrate ability and competence, resulting in better performance levels. These goals can also be combined with learning goals, favoring the engagement in self-regulated learning, which points to the importance of multiple goals in motivation and performance in educational settings (Cordeiro, Lens, & Bidarra, 2009; Lemos, 2005).

Moreover students‘ goal orientation varies from one person to another and with different situational demands, being triggered by specific achievement situations, in accordance with the goals emphasized (Kaplan, Middleton, Urdan, & Midgley, 2002). Different types of students’ goal orientation correspond, not only to different forms of individual motivation for learning, as well as ways to adapt to different goal structures highlighted by the school and the teacher in the classroom. In effect, the teacher's goal orientation and classroom practices, influenced by the school culture itself, affect students' goal orientation, learning and performance in achievement contexts. Teachers with a learning-oriented goal structure develop instructional practices that promote students' adoption of that goal structure, as do those who promote a performance orientation, which will also influence students' performance goals (Cordeiro, Lens & Bidarra, 2009).

Teachers who promote a learning-oriented goal structure, contrasting somewhat with a performance orientation, conceive learning as an active process, offer diversified opportunities to demonstrate mastery, stimulate students to diverse and meaningful activities, value formative assessment practices and individual progress, making use of individualized feedback, requesting students' interaction and involvement.

In sum, the Achievement Goal Theory includes the school and teachers in the equation of students’ motivation and has the merit of marking a turn in the study of quantity to the quality of motivation, considering the differential motivation value of the content (mastery vs performance) of the students’ goals.

1.8.2 Causal attributions and outcome control expectancy

When referring to the belief system and personal expectancy of outcome control in achievement contexts, namely in the school context, we address a set of sociocognitive-motivational variables related to self-functioning, distinguishing causal attributions from outcome control expectancy. While the former refer to the causes to which we attribute our performances, the latter refer to judgements about the probability of obtaining certain reinforcement as a result of our behavior, constituting a priori judgements, involving the concepts of locus of control, self-efficacy and learned helplessness.

Although several authors in the field of attribution can be highlighted), following the inaugural or seminal work of Heider (1958), it is the work of Weiner on the causal attributions on success and failure at school that deserve particular emphasis here. Weiner (1986) followed Heider’s proposal, when considering that people attribute the result of an action to certain causal beliefs related to different causes that may be internal or external to the individual, and with the perception of its constancy over time. In this scope, Weiner (1986) distinguished three dimensions to analyze the causes of success and failure at school: locus of causality, stability and controllability.

In fact, performance may be attributed to internal or external causes, located on a continuum, which we also conceive as more or less stable and more or less controllable. Weiner (1986) considered that people attribute success and failure to the presence or absence of ability, effort, as internal causes, and to the difficulty of the task and luck, as external causes, having developed an attributional theory of motivation and emotion in achievement contexts, assuming the influence of these attributions on expectations performance in the future.

For failure, attributions to stable causes, such as the ability and difficulty of the task, give rise to lack of confidence and discouragement, negatively influencing future expectations of control. In the same way, confidence and security are underlined when successful situations are attributed to ability and effort. Emotions of shame, guilt, anger and pride are also associated with the perceived controllability of causes (Weiner, 1986). The relationships between ability and effort, as well as between effort and task difficulty, are already discussed by Heider (1958), in the interplay of personal and situational forces in the causal attribution processes.

In turn, the relationship between attributions and emotions is developed by other authors and other dimensions such as intentionality and globality are also identified by some authors, although Weiner's three-dimensional model is more accepted, even though this author includes this dimension in a specification of the attributional theory in the approach to learned helplessness (Weiner, 1986). In fact, the dimension of globality, identified in studies on learned helplessness (Abramson, Seligman & Teasdale, 1978), enables us to understand the generalisation of the feeling of helplessness, which is usually associated with it, by relating the attributional processes to the expectations of control over the outcomes.

The concept of locus of control was developed by Rotter (1966) in his social learning theory and predates the concept of locus of causality defined by Weiner (1986), referring to expectations of outcome control which, in turn, are influenced by causal attributions, more specifically by the stability dimension. The perceived stability of causes influences expectations of future performance.

The locus of control refers to expectations of internal and external control over reinforcers, the belief that the results we obtain are more or less dependent on our behavior, and may be classified on a continuum from extreme internality to extreme externality. It is about the probability, evaluated by the subject, that a certain reinforcement will occur as a function of a specific behavior on his part, in a given situation or set of situations. Internality or externality is not a personality trait, but may be seen as a predominant tendency of the individual, resulting from learned beliefs, that may vary according to different contexts.

Learned helplessness is precisely linked to the perception of uncontrollability, external locus of control and low expectations of success. It exists when an individual cannot master the situation, generalizing responses of helplessness to other situations (Seligman).

The concept of learned helplessness proposed by Seligman involves the absence of real or perceived control over outcomes and is associated with situations of depression. In 1978, Abramson, Seligman and Teasdale reformulated Seligman's work, using attribution theory. They proposed that people differed in how they classified negative experiences on three scales, from internal to external, stable to unstable, and from global to specific. They believed that people who were more likely to attribute negative events to internal, stable, and global causes were more likely to become depressed than those attributed things to causes at the other ends of the scales. Learned helplessness is related to the concept of self-efficacy, the individual's belief in their ability to achieve goals. Perceived personal efficacy refers to the belief that one is able to successfully perform the behavior required to produce a given result (Bandura, 1997).

The concept of self-efficacy was developed by Bandura (e.g. 1997), in the framework of his socio-cognitive theory, which conceives the interaction between personal, behavioral and environmental variables in explaining human functioning. The reciprocity of these influences suggests that action on one of these variables influences the others. Thus, self-efficacy influences and is influenced by behavioral and environmental variables. Students who feel more effective engage more in self-regulated learning and create more supportive environments for learning, and self-efficacy is in turn influenced by academic achievement and teacher feedback and comparison with peers. The concept of self-efficacy is thus related to the perceived ability to learn and to perform action at certain levels (1997), influencing activity choices, effort, persistence and performance (Shunk & DiBenedito, 2016). According to Shunk and DiBenedetto (2016), compared to students who doubt their abilities, those with high self-efficacy participate more readily, work harder, persist longer and show more interest in learning and perform at higher levels.

The sources of self-efficacy include previous achievements, vicarious experience, forms of social persuasion and emotional and physiological states, such as anxiety or stress. Although self-efficacy is important, it is not the only influence on behavior, as Bandura (1997) points out. Values are equally important in the sense of the perception of the importance and usefulness of learning, as well as the development of effective skills for learning and performing tasks. However, self-efficacy has several effects in the educational context, namely on motivation, self-regulation, learning and performance. It should be noted that students sometimes overestimate or underestimate their abilities, which affects their motivation and performance, although it should be noted that when the perception of self-efficacy is higher than performance, this is not unfavorable at all as it can result in greater motivation and engagement, which will have an effect on learning (Bandura, 1997, Shunk & DiBenedetto, 2016).

We can distinguish several types of self-efficacy, namely self efficacy for performance, for learning, for self-regulated learning, or a collective self-efficacy, when students work together to accomplish a task, including also teachers instructional self-efficacy and classroom management, and a collective self instructional efficacy (Bandura, 1997, Shunk & DiBenedetto, 2016).

It remains to be added that self-efficacy is a dynamic concept that is continually susceptible to change, as Schunk and DiBenedetto (2016) point out, and it is a challenge to account for this dynamic nature of self-efficacy and its implications for educational practice.

1.8.3 Academic engagement: conceptualization and levels

Associated with the study of motivation, the concept of student engagement has come to be seen as crucial to understanding academic performance. From a historical perspective, the desire to enhance student learning may be considered at the origins of interest in academic engagement, as the primary focus of engagement was on preventing school dropout and in school interventions to promote students' development in the academic domain (Reschly & Christenson, 2012). Initially, the focus of the conceptualisation of student engagement was centred on academic engaged time, but it was soon realized that an understanding of engagement processes implies the consideration of other dimensions. The interest and evolution of the research on student engagement was synthesized by Reschly & Christenson (2012, 3) when referring that 'engagement has long been viewed as more than academic engagement time. From the earliest review to include the term engagement to the publication of seminal theory about underpinnings of school dropout and completion and more recent conceptualizations, engagement is viewed as multidimensional, involving aspects of students' emotions, behavior (participation, academic learning time) and cognition'. In this scope, it is considered that engagement has a much wider scope than academic dimension (Reschly & Christenson, 2012, 3), 'In other words, academic engaged time is important but not enough to accomplish the goals of schooling – students learning across academic, social-emotional and behavioral domains'.

The characterization of the facets of students' engagement has been one of the main focuses of research on the subject, and it can even be said that the growing interest in the subject has also been associated with the increasing variability in how this construct has been conceptualized over time. Fredricks and McColskey (2012) present an overview about the evolution of student engagement concept, either at the level of the dimensions that characterize it, or at the level of the components of each of these dimensions, from a two-dimensional model to others that already consider four dimensions. According to this review, the two-dimensional model of engagement considered two key dimensions: behavior, which in turn comprises participation, effort, and positive conduct, and emotion, where are considered interest, belonging, value, and positive emotions.

In three-dimensional models, a cognitive dimension is also considered where aspects such as self-regulation, investment in learning or strategies used. There has been some consensus around the three dimensions of student engagement, but there is some diversity when analyzing the aspects considered in each dimension. A summary proposed by Fredricks and McColskey (2012) highlights this situation. Thus:

- behavioral engagement draws on the idea of participation and includes involvement in academic, social, or extracurricular activities and is considered crucial for achieving positive academic outcomes and preventing dropping out. behavioral engagement may also be defined in terms of positive conduct, such as following the rules or child's effective behavioral participation in their learning process.
- Emotional engagement focuses on the extent of positive (and negative) reactions to teachers, classmates, academics, or school and involves identification with the school and the feeling of belonging or of being important to the school, and valuing, or an appreciation of success in school-related outcomes.
- Cognitive engagement has been defined as a student's level of investment in learning, including dimensions as being thoughtful, strategic, and willing to exert the necessary effort for comprehension of complex ideas or mastery of difficult skills or the child's mental orientation during learning.

Other models have also considered additional dimensions. For example, Reschly & Christenson, 2012), conceptualized engagement as having four dimensions, adding the academic dimension to the behavioral, cognitive, and psychological (subsequently referred to as affective) dimensions. In this model, aspects of behavior are separated into two different components: academics, which includes time on task, credits earned, and homework completion, and behavior, which includes attendance, class participation, and extracurricular participation. Veiga et al. (2012, p. 7477) also proposed a model on student engagement with four dimensions. considering personal agency, a dimension to be added to the behavioral, affective and cognitive, and conceptualized 'as students' constructive contribution to the course of the instruction they receive'.

One of the topics under debate concerns the specificity or possible overlapping of the concepts of engagement and motivation. A review of definitions of the concept by Kong and Lam (2003, 6) shows that since the emergence of research on the concept, this has been considered complex, 'not simply a commitment to complete assigned tasks or to acquire symbols of high performance such as grades or social approval. It is not directly observable and is something more than motivation'. According to the review by Fredricks and McColskey (2012), engagement has an action dimension or the behavioral, emotional, and cognitive manifestations of motivation, whereas motivation focuses on the psychological processes associated with the reasons or motives underlying an action, and can be conceptualized as in terms of the direction, intensity, quality, and persistence of one's energies, whose conceptualisation has involved a set of constructs e.g., intrinsic motivation, goal theory, and expectancy-value models, some of which were already been mentioned, to try to understand possible dynamics around the reasons for action ‘Can I do this task’ and ‘Do I want to do this task and why?’

It has also been considered that the potential of the concept of engagement and what makes it 'unique' lies in the fact that it simultaneously encompasses these various dimensions, emerging as a ‘meta-construct that includes behavioral, emotional, and cognitive engagement’ (e.g. Fredricks &. McColskey, 2012). ‘Although there are large individual bodies of literature on behavioral (i.e., time on task), emotional (i.e., interest and value), and cognitive engagement (i.e., self-regulation and learning strategies), what makes engagement unique is its potential as a multidimensional or ‘meta’-construct that includes these three dimensions’ (Fredricks &. McColskey, 2012, p.).

On the other hand, it has also been considered that the involvement reflects an individual’s interaction with context, e.g. with a task, activity, relationship, involvement always implies consideration of a context, and cannot be separated from their environment. In this context, it is understood that engagement is considered responsive, reactive/malleable to variations in context, in line with interactionist models of school performance and socio-cognitive models of motivation that consider complex and subtle dynamics between diversified characteristics of the person, the context and the results themselves.

In summary, the concept of student engagement has emerged as highly complex and multifaceted. It can be seen that besides the variation in the number of subcomponents, including different conceptualisations, there may also emerge different contexts in which it may manifest itself: school, colleagues, teachers, family, etc. Its specificity seems to reside in this very complexity, giving it a unique potential, which allows it to become ‘the glue, or mediator, that links important contexts – home, school peers, and community – to students, and, in turn, to outcomes of interest’ (Reschly & Christenson, 2012, 3).

The promotion of student engagement is of particular importance when considering issues of learning or achievement in mathematics. Learner disengagement is a familiar problem for mathematics teachers and several studies and reports at international level refer to this situation (e. g. O'Brien, Makar, Fielding-Well, 2015). In this context, the interest to analyze characteristics of engagement in mathematics and to explore strategies that can promote it, in particular the methodology that has been developed within the PUNTE project.

1.8.4 Final considerations

In short, it is important to take motivation into account not only as a means but an end in itself, developing in students the ability to self-motivate, to orientate attributions of success and failure towards the effort made, reducing the competitive nature of the classroom, which highlights both success and failure, by stimulating attributions for abilities and orienting towards achievement goals centred on results, in contrast with orientation towards learning-centred goals.

On the other hand, it is important to understand that engagement reflects an individual’s interaction with context, e.g. with a task, activity, relationship, and cannot be separated from their environment, that engagement is responsive, reactive/malleable to variations in context. This is in line with interactionist models of school performance and socio-cognitive models of motivation that consider complex and subtle dynamics between diversified characteristics of the person, the context and the results themselves.

It can be mentioned finally that there is already evidence of the potential of so-called 'student-centred' methodologies, based on problem solving, inquiry pedagogy to promote students' engagement in mathematics, realizing also the interest and potential of the methodology that has been developed within the PUNTE project.

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1.9 Inclusion

The root of inclusion comes from our belief according to which, full participation in society is one of the basic human rights. Therefore, education has to be organized in such a way which provides possibilities for developing all children to their full potential and for preparing them for productive and happy life as an equal and accepted member of his/her community. The goal of schools is to create as favorable learning environment as possible for almost all children of the community from which its students come from. Keeping this in mind Peters (1999) gives his definition of inclusion as follows: 'Simply defined, it means including disabled students with non-disabled students in every aspect of education, from the same classrooms to the same social activities and support groups.' Inclusion according to Mag, Sinfield and Burns (2017) refers to'… those who may be potentially marginalized by learning need or social position' in a regular class is always applied when learning can be achieved at a satisfactory level with the help of an assistant and the use of various aids. Some believe that inclusion has to be applied not only when students with disabilities are in question, but for each student when he/she has some specific abilities and needs. In both cases, not only by supporting students' learning of subject materials but also working on their social and emotional development flexibility is very important:

- Curriculum has to be flexible.
- Teachers have to be flexible. They should use and construct different teaching methods, aids and learning materials and they also have to modify and adopt the existing materials.
- Schools have to be flexible and must be ready to make necessary changes in infrastructure and classrooms.

Due to these flexibilities, each student can explore and learn in a learning environment suitable for him/her. This means that while all the students are learning the same lesson, many of them are doing different things and using various tools. Regular teachers, as well as specialized educators, during their teacher training, have to learn a broad range of techniques, strategies and methods of teaching which can enable them to be flexible and to be able to help their students to overcome the students' specific learning difficulties. (Imaniah and Fitria, 2018; Peters, 1999; Mag, Sinfield and Burns, 2017) emphasize that: 'Inclusive education is one of the top challenges in today's world, and whilst educational systems make efforts to become more inclusive, new teachers must be developed to be more inclusive in their future practice.'

Social interactions has crucial role in inclusion and as Peters (1999) says: 'Inclusion leaves a great deal of time for social interactions. Nevertheless, those interactions are not harmful or disallowed; they are rather part of the curriculum.' The students help and support one another during learning, which is usually organized in collaborative groups or pairs. While they are working together and cooperating with one another, the members of a collaborative group develop their problem solving, communication, social and behavioral skills, their sense of empathy, respect for others and positive self-image.

In order to make inclusion more efficient, some students' have their personal assistants to help them interact, and communicate with other students. (Yell, 2012). Inclusion of families of the students into the whole process also improves it.

Inclusion is beneficial for all the students and teachers involved in this process. (Peters, 1999) All participants of this process gain a wide array of experiences. They increasingly accept differences and develop their mutual friendship. Therefore, inclusion in schools prepares all its participants for life in an inclusive society where people are treated fairly and have equal opportunities for their development.

In order to help instructors to organize and conduct inclusive learning, Center of Excellence in Learning and Teaching of Iowa University of Science and Technology came up with ten inclusive learning environment strategies and eight of them are listed here:

- Examine your assumptions,
- Learn and use students' names,
- Model inclusive language,
- Use multiple and diverse examples,
- Establish ground rules for interaction,
- Examine your curriculum,
- Strive to be fair,
- Be mindful of low ability cues.

More about these strategies can be read on the website mentioned in bibliography.

As a very useful tool Poly-Universe can enrich the inclusive learning environment and can be a supplementary aid which can help disabled students' exploration, reaching conclusions and their communication to one another. It can be used for development of collaboration, cooperation and making friendships among the members of small inclusive learning groups. Poly-Universe can be used to show students whose vision is excellent, what the world looks like in the sense of the visually impaired students. Namely, when all students have to close their eyes and play games using Poly-Universe sets, they will be in equal position with their visually impaired peers. These games can be used in teacher training to enable new teachers to be more tolerant and inclusive.

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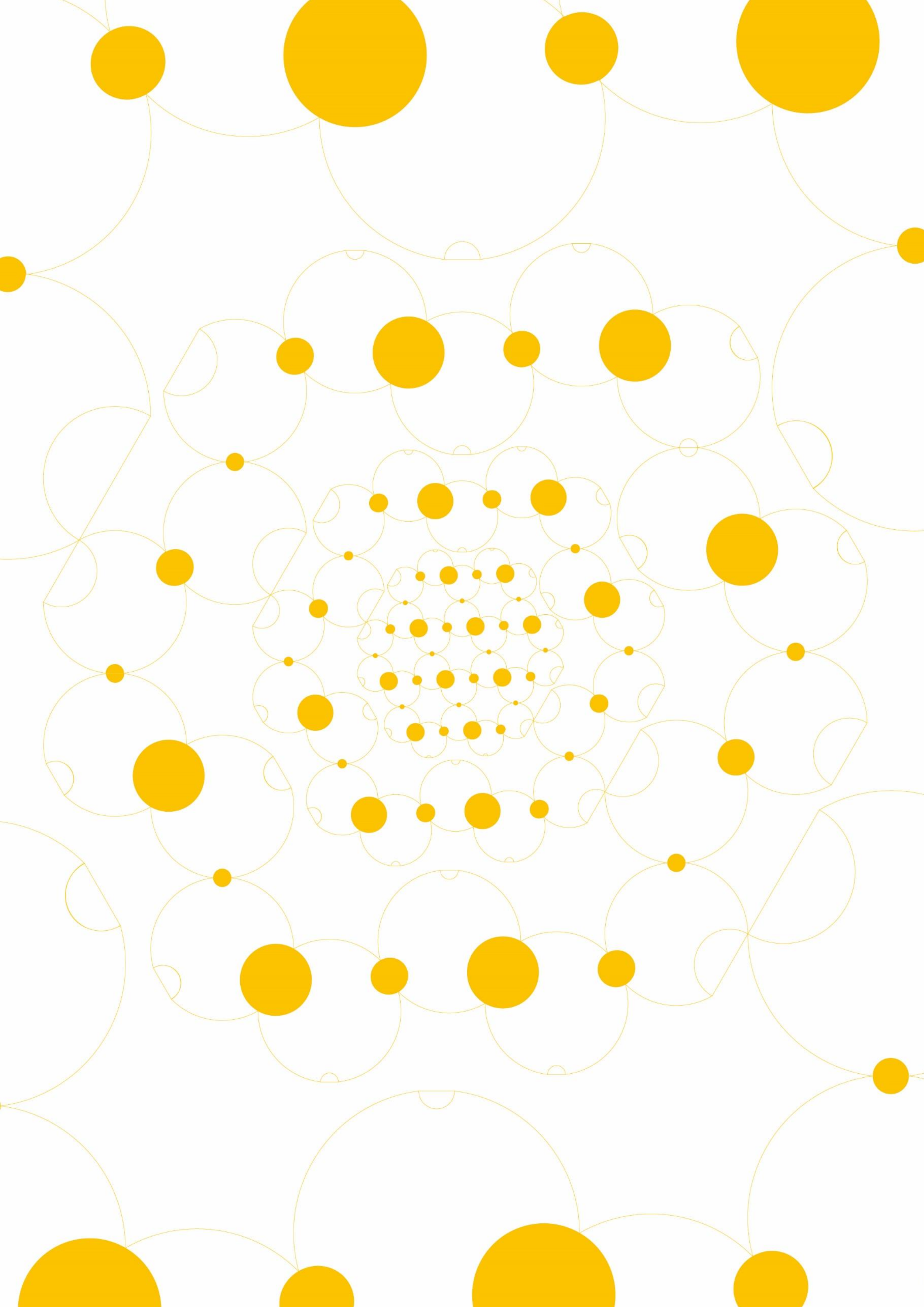


II LEARNING/TEACHING THROUGH ART

2.1 'Teaching to see' — Dimension change in geometric art, and education

'...Cannot to live without geometry: our Earth, the sun around which it revolves, the orbits of the planets, the bed we sleep in, the table we write on, the plates we eat on, the vase we pour water into, the utensils we use in the kitchen, are all geometric. The most wonderful organ, without which our civilization could not have come into being: the eye, a geometric shape, full of movement, full of dynamism. The eye is the object of light, said Leonardo da Vinci (Carmelo Arden Quin, founder of MADI movement) [1]'.

The inventor of Poly-Universe Game, János Szász Saxon visual artist, like the scientist, is a researcher, an explorer of unknown areas of reality for humanity. Communicates his knowledge of the fundamental nature of the Universe through his geometric works, paintings and sculptures. His artistic career has been going on for forty years. His vision is informed by his knowledge and experience as a child studying, the growth of fractal-like trees and leaves, and the movements of animals, insects and arthropods. Artistically, this world was at once cubistic, geometric and kinetic. A sensitivity to abstraction was therefore developed in him from a very early age. His ability to think abstractly, his love of mathematics and his achievements in this field could have led him naturally to the science of theoretical mathematics, to becoming a mathematician, but in high school an incurable eye disease cut him off from the outside world, so he concentrated on fine art. Today the two fields overlap, then it has to show in the field of education, or the art & sciences professional conferences.

At the beginning of the new millennium, the link between past and future, science and art, appeared on the horizon. It was timely, since Carl Jung had already created the archetype of the artist-scientist at the beginning of the 20th century; Albert Einstein saw both science and art as an answer to the mysteries of the world; not least the poet Károly Tamkó Sirató and the leading European artists of the time, who formulated their new understanding of the world in the Dimensionist Manifesto 1936 [2].

We are on our way. STEM (Science, Technology, Engineering, Mathematics), which has been key concept since the end of the last century and has been used in education worldwide, has nowadays been supplemented by the concept of Art, out of a thirst for completeness and to make up for a lack of creative creativity, and has become STEAM, which has become an international initiative in the last few years.

In addition to scientific research, STEAM has had a significant impact on creative work, design, interdisciplinary education and gamification, due to its modern approach to innovation. Within the interdisciplinary network of the new age, artists can work as equals with their scientific colleagues, and the two fields are once again coming together to reflect on the challenges of the 21st cent. [3].

The role models of the inventor of Poly-Universe, come from masters of geometric art, who were able to make dimensional leaps with their artistic work. The three of them are: Kasimir Malevich (suprematism), Carmelo Arden Quin (MADI), Károly Tamkó Sirató (Dimensionist Manifesto).

The fourth dimensional jumper is the inventor himself. In addition to personal creative work, these artists also did community work and dealt with children, pedagogy, as Carmelo Arden Quin said — 'There is a collective program and an individualist program in man, so individual and movement must be cultivated together.'

The Russian avant-garde artist Kasimir Malevich, was the creator of suprematism and the creator of the black square on a white background. In the following, I highlight the artists, and art history trends, and schools who influenced the inventor, and contributed to create the Poly-Universe tool.

2.2 Fundamentals of the geometric art in XXth Century with pedagogue correlations

2.2.1 Vitebsk People's Art School — Malevich's objectless world

Kazimir Malevich wanted to achieve a complete detachment from objectivity in the visual arts. He worked on the development of a pure artistic movement, free of ideas and associations. He painted one large motif on each of his canvases. On a white background he depicted black and then white squares; red or black crosses and circles. His reduced compositions were replaced by his suprematist works. In his 'supremus' paintings, he created a system of logical relationships between plane lines, lines and circles floating on a light background. The contemplation of these variations is thought-provoking, the viewer seeks regularity, while Malevich attempts to find harmony, balance, and proportion. He is interested in the structure of the image, the essence of its internal structure [4].

Figure 1: Kasimir Malevich: Black Square 1913 (State Tretyakov Gallery, Moscow)

Suprematism is not only 'the end of the number line, but also the beginning, like zero. It's the beginning of something that a constructive artist starts from this zero [5]'.

Malevich's pedagogical method was fulfilled in the Vitebsk School. The school was founded in 1919 by Chagall, El Lissitzky and Malevich. They promoted left-wing art in the spirit of collectivism, education and innovation. Anyone could enroll, there was no tuition fee and no age limit. All this fitted in perfectly with the ideals of the Russian Revolution.

With like-minded colleagues and students, he founded the UNOVIS (The Champions of the New Art) movement. Its members designed posters, magazines, flags, meal tickets, theatre sets, and trams. The small town was covered with colorful squares and circles, with the slogan 'the streets are our palettes' [6].

2.2.2 BAUHAUS — Constructivism as functional art

The two-dimensional possibilities of suprematism are further developed by constructivism. From planes he builds up masses, spaces. For the artist, creation is no longer just a logical construction of systems. The artist sees himself as an engineer. He observes the characteristics of the materials used by architects, such as wood, iron, stone, concrete, he constructs, builds, and creates using industrial processes.

Figure 2: The Bauhaus workshop building in Dessau (Photo: bauhaus-dessau.de)

The Weimar Bauhaus school, which was founded in 1919 by Walter Gropius, was a school related to the spirit of constructivism and was created to modernize architecture. His school of art was intended to be both a workshop for the fine and applied arts. The curriculum covered all artistic disciplines.

Technique and technology paved the way for industrial design. Furniture, irons, fountain pens and utensils were designed. Some moulded them in clay, others wove them into carpets.

In addition to sculptures in plaster, wood and bronze, machining workshops produced moving sculptures and mobiles that operated by electricity, emitting sound and light signals. Laszlo Moholy-Nagy combined moving metal alloys with lightweight, translucent Plexiglas in his Light-Space Modulators.

In addition to the physical workload, there was also time and space for theatrical experimentation, set and costume design. The Bauhaus theatre was born, and Oskar Schlemmer's experiments gave it a further boost. Photography and film together became a new medium in the Bauhaus programme. Learning artistic photography and realizing ideas for film provided students with a rapid experience of success.

Gropius moved the Bauhaus to Dessau in 1925, and in 1932 the school moved to Berlin. In 1937, the Bauhaus moved to Chicago.

Marcel Breuer, György Kepes and Laszlo Moholy-Nagy were also prominent figures in Hungarian art education, professors of the Bauhaus, and their art education principles were the precursors of the 21st century's summarizing concept STEAM. As György Kepes put it — we must have the brain of the scientist, the eye of the painter and the heart of the poet at the same time. [7]

2.2.3 Fajó school — Constructivism as aesthetic art

The school of János Fajó (1976–2016) carried on the spirit of the Bauhaus, the intellectual heritage of Kassák, Moholy-Nagy, and Bauhaus. The school's art teachers taught painting and sculpture, photography, design, ceramics and architecture, teaching pure form in all materials, techniques, theory and practice [8].

Figure 3: János Fajó: Plane Painting school book (The way – The method)

The inventor of Poly-Universe tools, Saxon at the age of fourteen was sent to a Constructivist-oriented free school, where he worked in the sculpture class of Tibor Csiky and the painting class of János Fajó. He attended the free school for five years, where he learned the formal language of geometry from his masters, and it was these years that laid the foundations for his commitment to a career in the fine arts.

Figure 4: 'The art of pure form Homage to Kassák' exhibition of the Fajó School, MÚOSZ Budapest, 2013

Let's listen to a thought from him, about how the concentrated presence at the School of Fajó affected his personality development in the art:

'I was sculpting by day and painting by night. Sometimes for three days without stopping until I tumbled down. I was pushing the envelope of my physical capacity. Although the body needed a rest, the mind wanted to soar and could throw off the shackles of the body. I acquired the form of expression of geometric art in this summer school from my mentors and these years established my devotion to the career of an artist. Since I could not find the secret of the point in mathematics, I turned my attention to art. During the course I made my first graphic work of art called 'Universe'. The picture demonstrates the possible permutations of halving the diagonals of the square. This work contains all the previous observations and knowledge and gives you the pure feeling of the Universe. Then I did not think how important this work, which did not really fit into the works made there, would be in the future. I returned to that school every summer for years and created Constructivist, Formalist works there, due to the effects the course had on me. After a while, finding and using geometric shapes and colors in my compositions became a routine and took place on the surface. Like copying nature, it could not really hold my attention any more [9].'

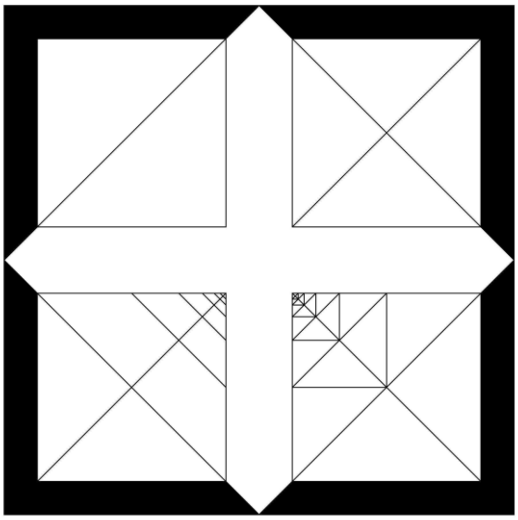


Figure 5: The first Saxon artwork — Universe 1979, ink, paper, 47x47 cm

It was not easy to break away from the influence of his masters and find his own style. But with his first painting, 'The Universe', he was interested in the deeper context, the structure of the Universe, its principles and the ways in which art could represent it. Thus came 'Structure' (1984–1988), which he built up over five years, and the Star series (1984), which focused on the connection and unity of the vast and the infinite. This was indeed the end of his first constructivist era. After these, his paintings went beyond the physical world, as they were not compositions but spiritual and mental projections. He called this period 'intuition abstractions'.

The last painting of this period is a white cross embedded in a disintegrating yellow circle, 'The Cross' (1989). This work, the contrast of yellow and white, the mystery of being and non-being, the transcendent meaning of form, was soothing. He used these two colors in the 1990s because he felt that yellow contrasted more vividly with white to reflect a sense of being and non-being, of something and nothingness, than say black and white.

2.2.4 Ateliers Pédagogiques — Espace de l'Art Concret (EAC)

With Gottfried Honegger's school in mind, 'Poly-Universe' was created in Mouans Sartoux (F), in the context of the Espace de l'Art Concret (EAC) pedagogical workshop, in 2000, in the framework of a six-month fellowship.

Since its creation in 1990, EAC's primary mission has been to make contemporary art accessible to the widest possible audience. Art education has always been at the heart of this mission, with an entire building dedicated to studios designed to host groups of children from kindergarten through high school. The pedagogical objective of Gottfried Honegger, a concrete geometric sculptor, designer and museum founder, who created the art scholarship, was the idea of 'Teaching to See' — *'Vision is our most important sense. Our brain thinks in images. It remembers what it has experienced. Our memories, our dreams are images. The quality of the programs memorised in the brain determines the quality of our thoughts, emotions, and actions. So it makes a difference what kind of vision children are confronted with from a young age to adulthood... and beyond [10].'*

Figure 6: Espace de l'Art Concret — Ateliers Pédagogiques, France (on photo Zsuzsa Dárdai 2000)

Honegger invited Saxon to join the local pedagogical workshop during the fellowship and to pass on his artistic vision to the children. After six months of working in a real studio for the first time in his life, he wondered how he could convey the essence of his art to the children. To this end, he began to systematize his painting methods, experimenting with forms and following the laws of basic shapes. Thus he created polygonal, free-form, geometric constructions, pulsating between the micro and macro worlds, what he called 'Poly-dimensional Fields'. He realized that the system was actually constructing itself.

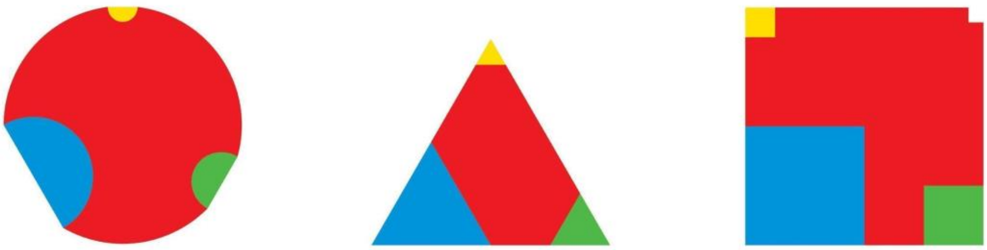


Figure 7: Saxon — Poly-Universe game basic elements (circle, triangle, square)

This is how the first playful elements of the Poly-Universe were created.

While he was working in the collision zone between concrete art and MADI, in the citadel of concrete art, Saxon was already busy organising the events of the MADI international art movement in Central and Eastern Europe.

2.3 Art Concrete and MADI movement

The term concrete art originated with Theo van Duisburg (Art Concrete, 1930), although after World War II Kandinsky tried to replace (clarify) the concept of abstraction with this term, and Max Bill built a school around it. Bill used the term in the Duisburgian sense, as the Abstraction-Creation group in Paris did between 1931 and 1936: he understood it as abstraction rejecting all forms of expressionism, an approach rooted in constructivism and suprematism.

Figure 8: Max Bill — Color field with white and black accents 1964–1966 (Zurich)

The principles of concrete art [11]

1. Art is universal.
2. A work of art must be conceived and formed entirely by the mind before it is executed. It can get nothing from the formal data of nature, sensuality or sentimentalism. It wants to exclude lyricism, drama, symbolism and so on. The painting must be built up from purely plastic elements, namely surfaces and colors. A pictorial element has no meaning other than 'itself'; consequently, a painting has no meaning other than 'itself'.
3. The structure and elements of a painting must be simple and visually verifiable.
4. The painting technique must be mechanical, i.e. precise, anti-impressionistic.
5. Striving for absolute clarity is mandatory.

What is the difference between concrete art and MADI?

Figure 9: Catalogue of the movement MADI int. exhibition, Maison de L'Amérique Latine, Paris 2008 (design Saxon)

The concrete artists work within the frame, on the stretched canvas, while the MADI artists break down the frame, free the form from the confines of the rectangle and open a path to infinity. The most important characteristic of MADI, then, is the polygonality with which Saxon was able to renew his work. In the 1980s, he painted his multidimensional planar structures on square canvas. But the infinite structures were trapped in the rectangle and 'wept bitterly'. Theoretically the theory 'Bildarchitektur — Architecture of the Image' of Lajos Kassák held out the possibility of free-form geometry, but in their materiality they remained within the square. The early Russian Constructivist masters such as Tatlin, El Lissitzky or the Hungarian-born László Péri had already created free form in the visual arts in an objective way.

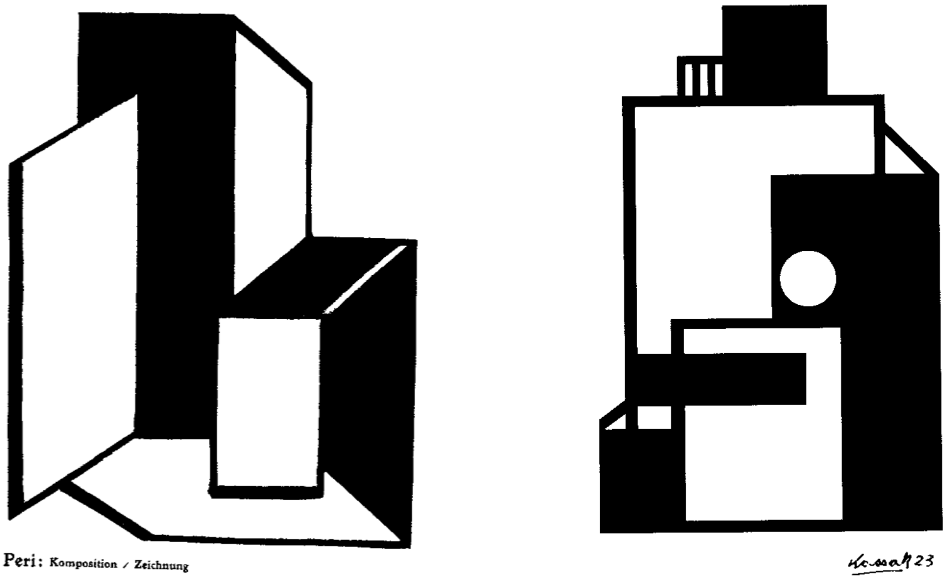


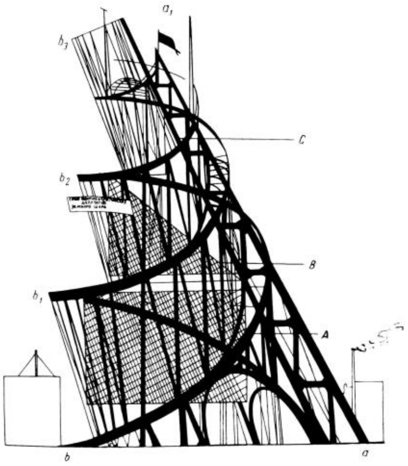
Figure 10: László Péri — Komposition 1923, and Lajos Kassák — Bild Architektur, 1923

In created of Poly-Universe tool, for Saxon the MADI was decisive, because its polygonality breaking into infinity, created the free-form pictorial object in geometric art, both theoretically and practically, and helped him to complete the Poly-dimensional Fields from the early nineties onwards, and had a major influence on the formal design of the Poly-Universe elements.

The MADI movement was founded in Buenos Aires in 1946. Its founders are Carmelo Arden Quin, Martin and Ignacio Blaszko, Estaban Eitler and Gyula Kosice. The famous Uruguayan artist and educator Joachin Torres García, who returned home after his years in Paris, played a major role in its creation.

2.3.1 Constructive Universalism

Torres García was one of the intellectual fathers of the Cercle et Carré art movement and magazine in Paris in 1930. Mondrian, Arp, Herbin, Vantongerloo, Kandinsky and himself formed an interesting and important group. He returned to Montevideo in 1934 and continued to build 'constructive universalism' in Uruguay. In 1935, Arden Quin, who was living in Buenos Aires, met Torres García at a conference on the European avant-garde. Through him they had access to art, books and newspapers. Arden Quin saw Moholy-Nagy's fantastic light-machine in one, the Futurists' manifestation of Tatlin and his tower in another, László Péri's moulded paintings in very poor reproductions, and Rodchenko's hanging sculptures. All this was a revelation for him. He had previously studied cubism until one day he found a cut-out polygonal form that someone had applied to a square base. He removed it from its base and held in his hand an irregular polygonal form that took its place in space. This gave him the idea to go on to create shaped cubist images and objects.



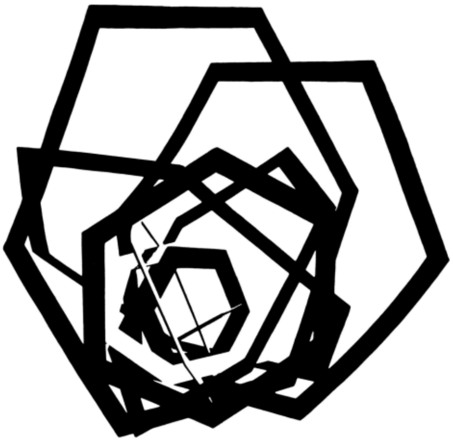


Figure 11: Vladimir Tatlin — Model of the Monument 3rd International, 1919–1920,
on right Alexander Rodchenko — Construction, 1920.

He showed his ideas to Torres García, who produced his famous geometric fish painting, made around 1930: a fish made of cut-out elementary shapes, with a hole for an eye. This is how Arden Quin came to know Torres-García's universal constructivism.

Universal constructivism (sometimes called constructive universalism) was an artistic style created and developed by Joaquín Torres García. He took the principles of Constructivism developed by Russian artists in the 1920s, which influenced the De Stijl and Bauhaus movements, and integrated them into the world of universal pictograms he created: the sun, the moon, man and woman... etc. His symbols refer to man, knowledge, science and the city, to life as a whole. This is how he develops his constructive universalism. [12]

Figure 12: Joachim Torres García — Constucción 1944, Catalogue Raisonné

While Arden Quin and the other founders of MADI were greatly influenced by Torres García's works and artistic ideas, MADI art was designed to transcend constructive, geometric, concrete principles and practices.

2.3.2 Dimension leap, the MADI principles

To understand this, we need to take a brief detour into art history. MADI has been formed by Impressionism, Cubism, Fauvism, Futurism, Dada, Surrealism, Russian suprematism, and Constructivism, it is a continuation of their heritage. If you analyze those movements, you'll see that each moved along a concept: Fauvism went back to the break with the first, direct view of nature; Cubism searched for the solid structure of sight and cut motifs into geometric forms, triangles, rectangles, and circles; the concept of speed, dynamism, and movement belonged to Futurism, we can say the first mobile was made by Futurists; Dada was provocation, negation, but also the absurd, the free verse that became the basis for the dream-automatism concept of Surrealism; the objectless world has come through the suprematism of Malevich, leading to the unfolding of Constructivism, but of course, it all began with Impressionism: it was Impressionism that put an end to academic rule and opened art towards space, towards the air of freedom.

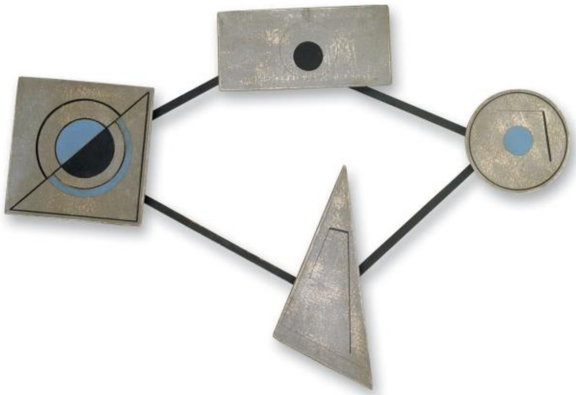


Figure 13: Carmelo Arden Quin — Coplanal 1946 (cooperation with planes)

These are great facts, but in the study of art history we never encounter the problems of polygonality. The word as a mathematical/geometric concept was familiar to artists of the time, and polygons even appeared as a compositional element on a rectangular surface, but only within the frame. Why were triangles, pentagons and heptagons not used? This change of dimension was missing in art, which is why the MADI form had to appear: to free the surface of painting from the rectangular frame in which it had been enclosed for centuries.

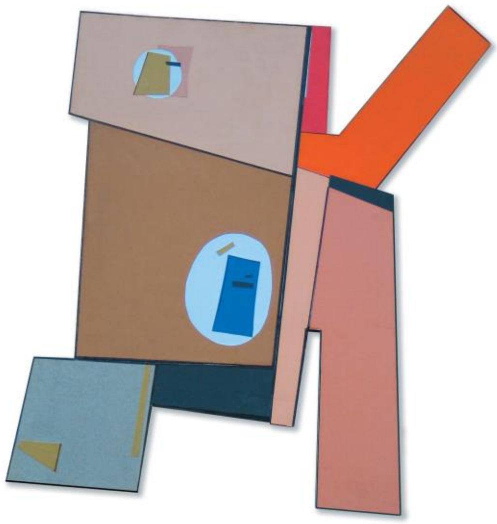


Figure 13: Martin Blaszko — Paris 1990, paper collage, wood, 55×59 cm (Mobile MADI Museum)

Carmelo Arden Quin moved to Paris in October 1948 and founded his MADI group there also, which was joined by Hungarian geometric artists in the early nineties, inspired by Saxon, and the international Mobile MADI Museum was established, followed by exhibitions, events and the publication of a journal. [13]

2.4 'Let it play' — The role of geometry in education

In the life of an adult, different forms of activity occur simultaneously, but not with equal importance. Children, however, acquire the different activities gradually: they are not yet working but already learning, not yet learning but already playing.

The use of games in school education – and in education of all kinds – is not a new phenomenon; it has been recognized since antiquity that the interposition of play activities can make teaching and learning more effective. In addition to Plato, who argued that the use of games with rules played a major role in the education of law-abiding people, Aristotle also mentioned the importance of play, recommending in particular the teaching of games to children that they would take seriously as adults. And the Latin Quintilian saw in play an unparalleled opportunity to learn about human nature. A few centuries later, simulation war games were used to train soldiers in the Prussian army; and in Victorian England schools used playful elements to teach geography, mainly in the form of puzzles.

Fröbel, who was known as a Pestalozzi pupil, also saw play as an exceptional educational opportunity, and he himself created a whole system of games and playful activities for young children, especially preschoolers. (The Hungarian Áron Kiss did the same with his collection of toys.) Freud, the founder of psychoanalysis, considered play to be one of the most effective methods of treating nervous children because, as he explained, play gives the child the opportunity to express his or her emotions openly. Psychologists have used this feature ever since for diagnostic purposes.

According to Huizinga, who studied the characteristics of homo ludens, play is in fact an older phenomenon than culture, because – and here he was certainly referring to group games – play is a manifestation of the struggle and hostility restrained by friendship. According to the play theory developed by the Swiss psychologist Piaget, the development of play activity can be seen as a specific explanation of the child's intellectual development.

These few examples show that the educational role of play is now well known from centuries of experience. This is why play pedagogy is now taught in all educational establishments with nursery school teachers, primary school teachers and teachers; there is a vast literature on the subject and the teachers concerned can learn about the educational value of the various types of play – movement, skill, development, social, etc. – not only from textbooks, but also through national and international professional events, conferences and campaign. [14]

2.4.1 Play-learning theory of Maria Montessori

I will focus on Maria Montessori's play-learning theory, because I feel it is close to Saxon's Poly-Universe.

For over a century, the child-centred approach to educating children developed by Italian physician Maria Montessori (1870–1952) has been transforming schools around the world. Montessori classrooms are instantly recognisable. Children work independently and in groups, often with specially designed curricula; they are deeply engaged in their work; they respect themselves and their environment.

Figure 15: Tools of Montessori from Westside Montessori School, Vancouver, Canada

Montessori education is essentially a model of human development and an educational approach based on this model. The model has two basic principles. First, children and developing adults engage in psychological self-construction through interaction with their environment. Second, children, especially under the age of six, follow an innate path of psychological development. Based on his observations, Montessori believed that children who are free to choose and act in an environment prepared according to his model will act spontaneously for optimal development.

The five basic principles for implementing the Montessori Method are:

1. Respect for the child: teachers show respect for children by helping them to do things and learn for themselves. When children are given choices, they are able to develop the skills and abilities necessary for effective learning, independence and positive self-esteem.
2. The receptive mind: Simply by living on, the child learns to speak his mother tongue.
3. Sensitive periods: It is a temporary predisposition and refers to the acquisition of a particular trait. Once this trait or characteristic is acquired, the particular sensitivity disappears.
4. The prepared environment: the prepared environment makes learning materials and experiences available to children in an orderly way. Freedom is an essential characteristic of the prepared environment. Because children are free to explore materials of their own choosing in the environment, they absorb what they find there.
5. Self-education. Children who actively participate in the prepared environment and who exercise freedom of choice are literally educating themselves. [15]

Before moving on to an assessment of the Poly-Universe, we would like to present some exciting examples from abroad.

2.4.2 Les trois ours

Since 1988, the ‘Three Bears’ publishing house has been tracing the history of geometric art and illustration in the 20th and 21st centuries to shed light on what the great artists thought and imagined for children. He publishes books, toys and artworks in numbered, one-off editions.

Goldilocks, the adventurous little girl, roams the house of the three bears, exploring, experimenting and choosing what is 'just right' for her. It was this spirit that was embraced by a founding group of graphic designers, artists and volunteers when the Three Bears Association was born. Year after year, they organize exhibitions and build relationships with geometric artists whose work is the perfect pedagogical tool to reach children through arts, crafts and play. [16]

From a rich selection of artists, here we present three artists closely related to our theme.

2.4.3 Gottfried Honegger

In 1993 'Le Viseur', a teaching tool for 'teaching to see' was designed. The toy, made up of simple and colorful geometric shapes, brings concrete art within the reach of children. Children of any age can walk through the garden of geometry. At their own will and at their own pace, they can travel from reality to imagination by inventing the rules of the game through the complexity of color, shape, rhythm, balance or randomness. The reduced quantity of shapes and colors allows the creation of simple and universal visual expressions.

This educational tool is often used in the context of various projects designed by the Pedagogical Workshops of Espace de l'Art Concret (EAC).

2.4.4 Bruno Munari

Bruno Munari designed the ‘Abitacolo’ in 1971. This modular and multifunctional structure allows children to read, work, play, organize and, above all, create and dream!

Maximum functionality in minimum space. Abitacolo was awarded the prestigious Compasso d'Oro prize in 1979. Everything in Abitacolo is modular, adaptable, adaptable to the needs of the child, stimulating their imagination, developing their autonomy.



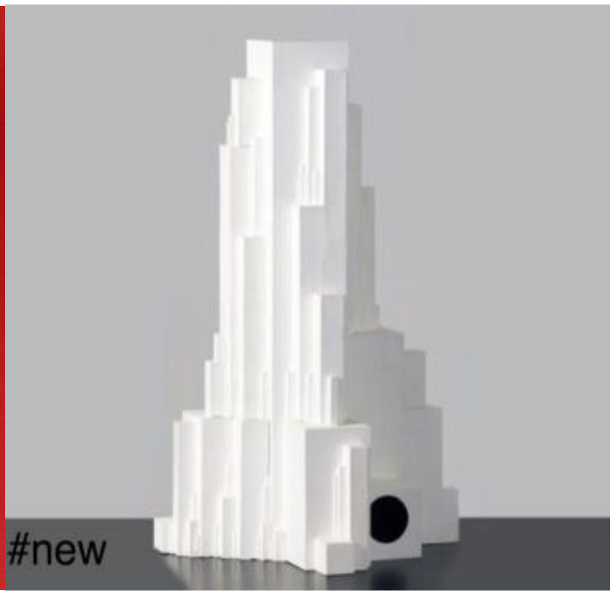
In the game ‘Plus-minus’, Bruno Munari offers transparent, opaque and perforated cards to children, who place geometric shapes on the cards, connect them to each other, and invent and tell stories, alone or in groups.


Munari has been exploring the interactivity of light and space with children through thousands of workshops since 1950.

During the workshop ‘Roses in salad: vegetable footnotes’, the children will make prints of cabbage, leeks and other vegetables, abstract landscapes and faces. The session will end with a festive lunch, during which they will eat the ‘tools’ of the workshop.

2.4.5 BeAMalevich

A Barcelona-based team of art, architecture and design enthusiasts working to create objects to arouse people's aesthetic and cultural curiosity. European artists and abstract movements of the 20th century mainly inspire them. They are in contact with the most important museums, cultural institutions, art and design shops and brand stores around the world.



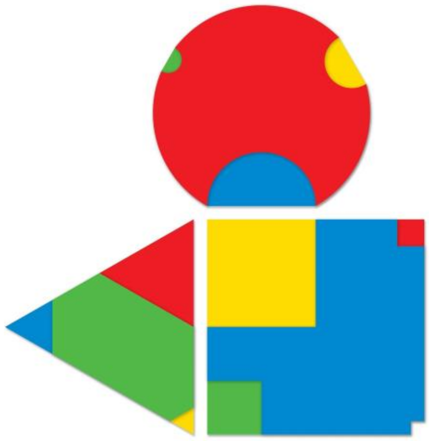
Behind BeaMalevich is the story of Xavier Vidal, an entrepreneur from Barcelona who, after living four years in Moscow, one day stood in awe before Kasimir Malevich's Black Square displayed at the Tretyakov Gallery. Xavier did not interpret that square as a dark hole, but as the threshold into an unknown world. Today, BeaMalevich is a company that develops and distributes objects inspired in art, architecture and creativity [17].

2.4.6 Saxon's Poly-Universe

The game family, in using and developing the basic geometric shapes of János Szász Saxon's poly-dimensional plain painting, communicates a new artistic perspective to both nursery and primary school children and adults. Having a direct, by-touch connection with the geometric shapes, the sense of vision and of touch are developing and through the recognition of correlations and finding the linkage points, the ability of thinking improves and the skill of abstraction evolves.

The learning process:

- Sight
- Touch
- Detection
- Perception
- Memory
- Attention
- Concentration
- The discovery of part-whole
- Imagination
- Creativeness
- Problem-solving



These colorful and easy-to-handle geometric shapes generate infinite logical complexity and complicated mathematical and morphological puzzles. However, its strength is its simplicity – it provides an equal opportunity for children of different ages, at different levels of mental and emotional maturity to develop their personality.

Figure 16: Workshop in Piarist High School, Vác, Hungary, 2016 (Saxon with students)

Progressive use of color and shape groups and the high degree of manual activity and reflective thought create a constant challenge for the children and maintain their desire for exploration, at the same time allowing an undisturbed and continuous feeling of success. Direct physical activity, the emotions conveyed by colors, and the possibility of trying out plenty of variations without being checked make the children feel free and relaxed. Such experiences help the children to find more creative and imaginative solutions when dealing with problems from different areas of life or to acquire new knowledge.

Thinking operations development:

- Analysis
- Synthesis
- Abstraction
- Comparison
- Perception of correlations
- Generalization
- Clarifying
- Analogy
- Order

All in all, Poly-Universe should not be limited to school-lessons, to after-school problem-solving courses, to any controlled activity (while being suitable for all these), but can act as a catalyst for the new pedagogical practice of learning by playing: teaching to see.

The game family did not set up any rules for the game. During the workshop the children and adults will not solve given mathematical problems, but they will recognize the mathematical and aesthetic correlations hidden in the system individually. These correlations could be summarized as follows:

a. Discovering geometric shapes
b. Searching for proportions
c. Examining symmetry
d. Finding the linkage points
e. Setting the directions
f. Making colors collide
g. Mixing forms
h. Expanding the limits of composition
i. Possibilities of combination
j. Feeling the infinite

So, this game family does not only aim at problem-solving or recognizing colors or shapes, or solving logical puzzles, but also offers the possibility of playing a game freely, so children or adults can learn indirectly, through a game, an activity [18].

Figure 17: Workshop with scientists in International Symmetry Festival, Wien, 2016

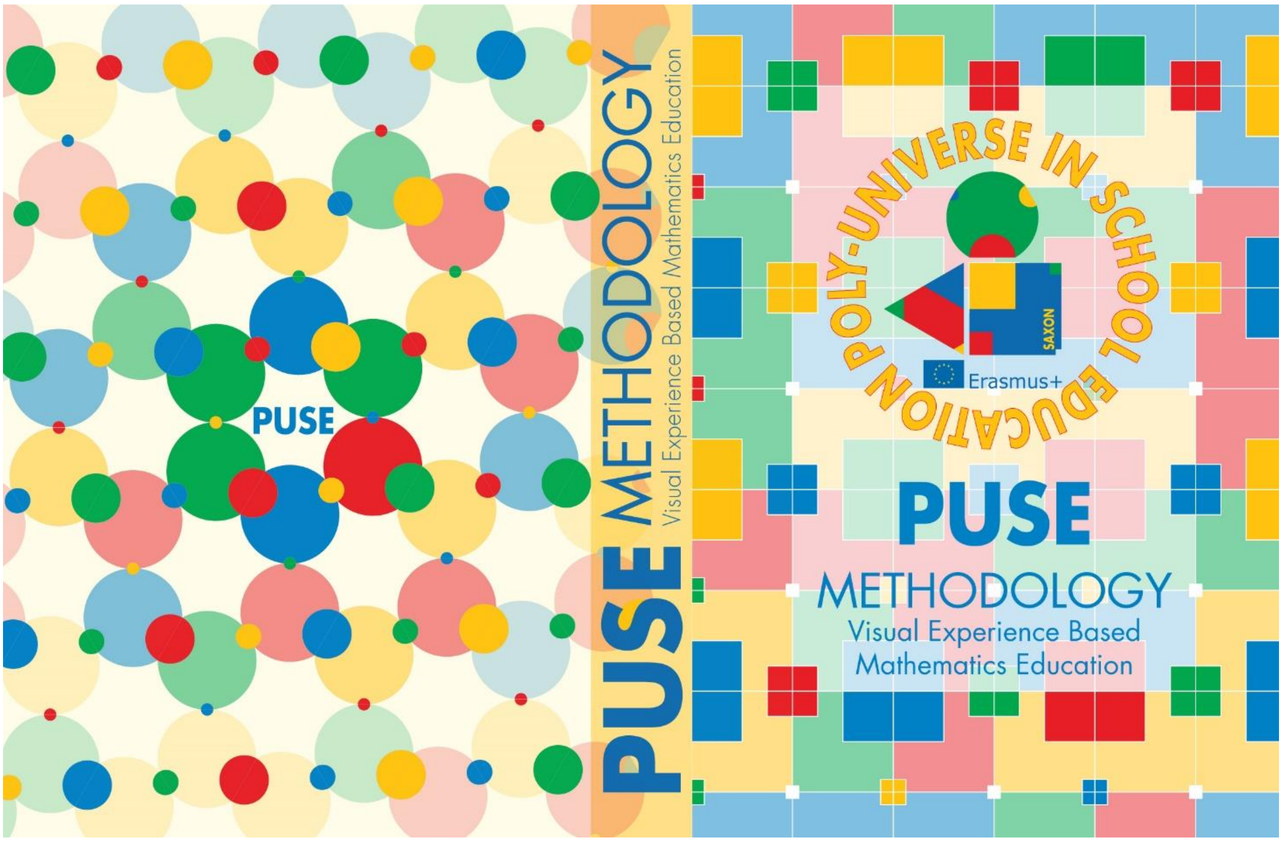


Figure 18: *PUSE (Poly-Universe in School Education) METHODOLOGY – Visual Experience Based Mathematics Education*

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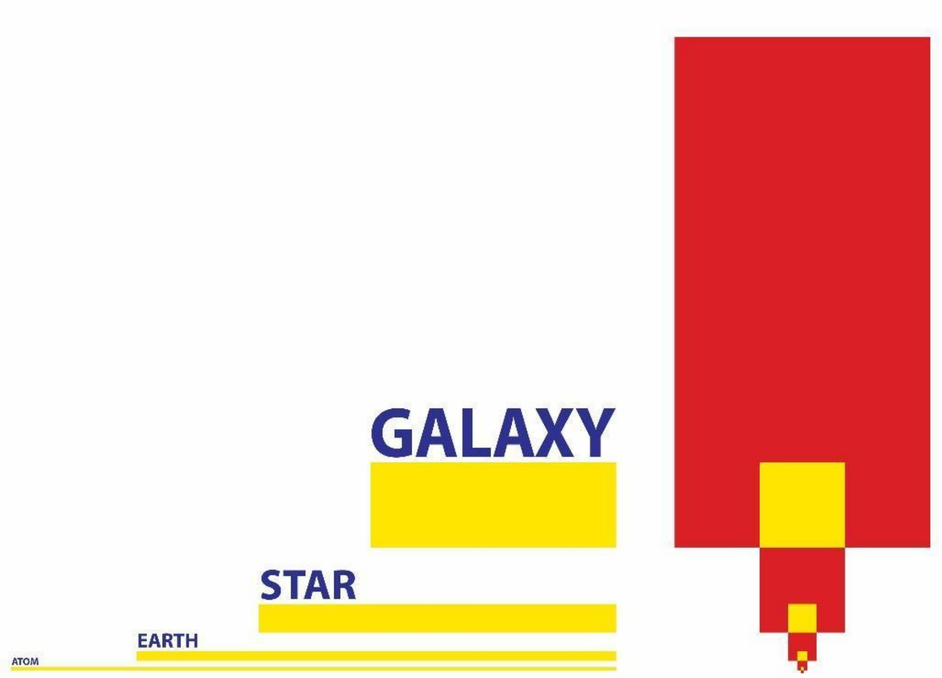
2.5 Dimension Pencil – as an imaginary tool for changing pedagogical approach

With the help of the dimension pencil, we would like to invite the reader to a spiritual excursion in the Poly-Universe, which pervades, and at the same time embraces us. Departing from the simple yet magnificent point, we will proceed through geometrical figures (circle–triangle–quadrangle), and eventually arrive at more complex spatial structures on our journey between dimensional structures on various scales.

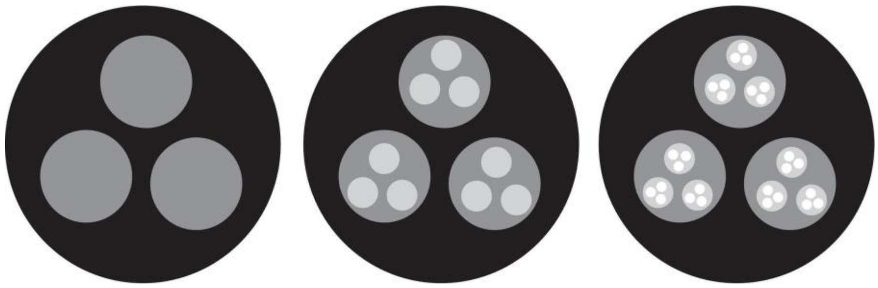


Our track will, of course, be determined by the map of the Poly-Universe: with our dimension pencil, we will simply draw a line between the unshakeable Galaxies, on the surface of our roamable Earth and around the buzzing Atoms. This fantastic fantasy may incite us to find our real place in the real universe. With the help of the dimension pencil, we can investigate the invariable connections between large, small and even smaller things, both in the living or 'inanimate' organisms and in the infrastructural growth of human society. Of course, taking these three steps is far from the end of the great journey.

Figure 1: Workshop with children in Espace de l'Art Concret, Ateliers Pédagogiques, France 2000

2.5.1 The 'poly-dimensional' point

From a mathematical perspective, we can say about a point that it is the smallest unit, a fundamental notion. On the other hand, these infinitesimal points which do not even have an extension constitute lines, planes, space, our physical world, and our infinitely large Universe as well. This is the real dimension paradox. We could represent this with a hierarchical model of the world, in which the lower-level systems combine to form higher-level systems.

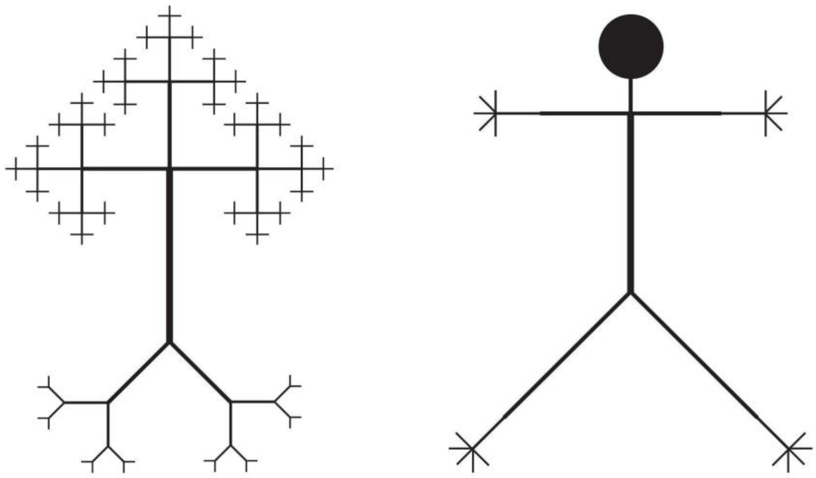


From this point on, it is merely a question of agreement between universes whether we can consider the atomic particle a point compared to the globe, the globe in turn compared to the Milky Way, the Milky Way compared to the immeasurable worlds built up of sets of Galaxies, or, to take a more tangible example, the (inseminated) ovum compared to a human being. Should this agreement be reached, we could define the point as a multidimensional phenomenon, as the condensation of all dimensions and dimension structures. The point, indeed, 'remembers' all dimensions and dimension structures: it is the intersection of lines, the micro-plane element of planes, the basic particle of space and in fact the impression of worlds on various scales.

Figure 2: Saxon — Poly-dimensional point 1990, oil, wooden board, Ø 152 cm

2.5.2 The ‘poly-dimensional’ line

All of us have probably already observed that the trunk of a tree branches in two or three directions, the thicker branches in turn divide into boughs of smaller circumference, down to the thinnest twigs at the end of which we can find the leaves. If we continue our observation, we may see that the capillary vessels within the leaves reflect the image of a small tree. Taking our contemplation even further, we might conclude that the divisions of our own body resemble those of the tree–the limbs (boughs) extending from the trunk end in fingers (twigs). Moreover, the network of veins in our bodies (or, for that matter, the network of fountains, streams and rivers all over the earth) is characterized by the same divisions.

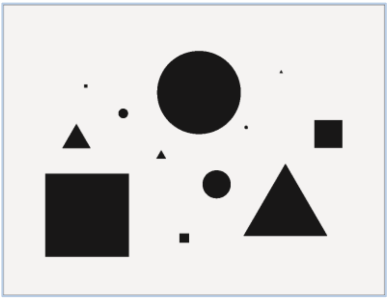


Of course, the divisibility of trees does not stop at the level of their capillary vessels; it carries on in the flow of molecular and atomic particles: the vital energy itself is radiated to the leaves straight from the star called Sun in the form of light. This is how the smallest and the largest we are capable of perceiving are connected: the worlds of atoms and stars in relation to a tree–and, evidently, in relation to us, too.

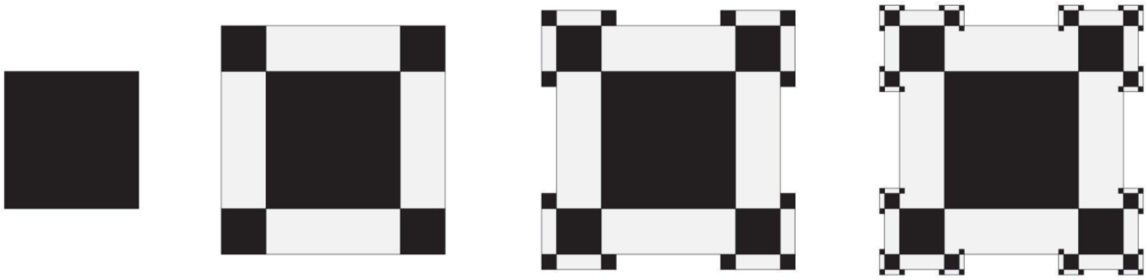
Figure 3: Tree and leaf examination — workshop with children in a school garden, Nagyvarsány Primary School, Hungary 2012

2.5.3 The ‘poly-dimensional’ plane

If we place geometrical elements of varying size or proportion, but of similar form, on a sheet of paper, our eyes will perceive the connections between large, small and even smaller elements in perspective.

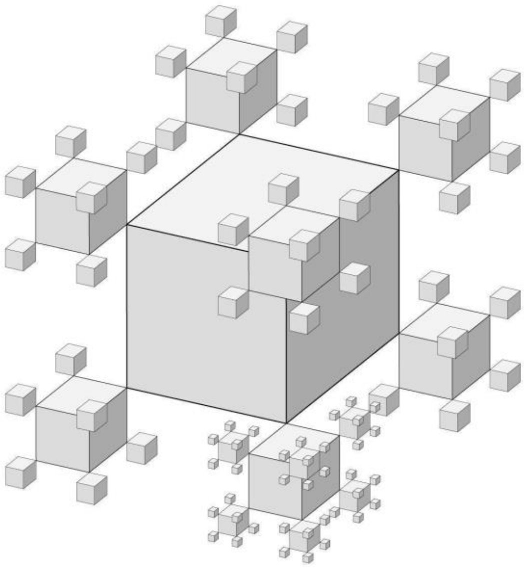


If, however, we connect and combine the same forms, perspective ceases to be effective, and we arrive at new structures constituted by the different forms attached to one another.



The ‘poly-dimensional planes’ thus emerging are able to model the abundance of nature (trees, blood and water systems, crystals, cell division, etc.) and the infrastructural growth of human civilization (networks of roads, pipe systems, networks of communication, etc.). On the other hand, they can represent the dimension structures of atomic and stellar systems, which have a similar structure, but are realized on extreme scales. As a matter of fact, this excessively visual attitude in art can be considered at least as ‘nature-based’ as Flemish landscape painting.

2.5.4 The ‘poly-dimensional’ space



In order to have an insight into the interconnectedness of spaces or dimension structures on various levels, let us place identically formed chess figures on a poly-dimensional field consisting of squares. The proportion of the figures should follow the different sizes of the planes. Let us now construct two stools of sixteen or sixty-four legs each, through the splitting of the legs accordingly. We can play mentally, or in real space with real figures.

Figure 4: Workshop with children in Espace de l'Art Concret, Ateliers Pédagogiques, France 2000

After the completion of Dimension Chess (1998), sitting down on one of the dimension stools, one will soon find out that the other stool must inevitably stay empty. This game is not a battle between two players; it takes place between one person and the spaces of unequal proportions. We ourselves are one of the figures lined up on the chessboard in front of us, and we can move as we please within the Poly-Universe opened up by the vertically linked spaces. With each move, we dispose of the parameters of the previous world.

In our lives, all of us make two such painful moves in the real world, too. The first one when, starting from an ovum, we are generated into human beings in the biomass (a process somewhat like the creation of the world, perhaps), and the second one at the end of our earthly life, when we rise back into the subtle regions of the spirit.

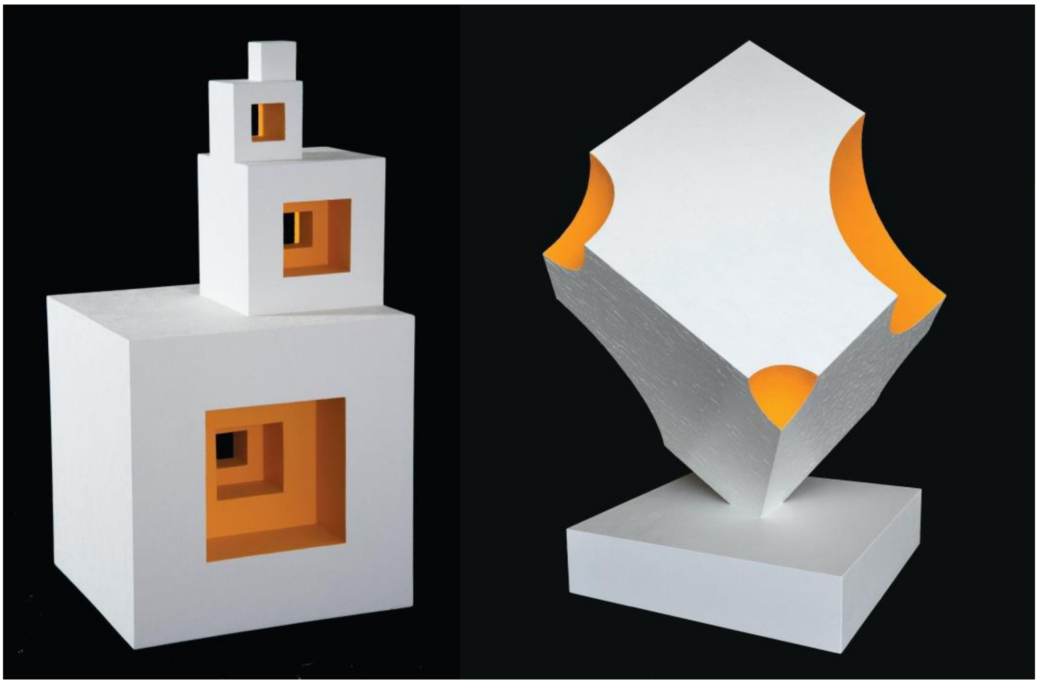


Figure 5: Saxon — Poly-dimensional tower with spaces 2015, oil, compressed wood, 32x30x60 cm
Saxon — Poly-dimensional Negative Planet 2015, oil, compressed wood, 60x50x75 cm

2.6 Basic geometric shapes: Square, triangle, circle

2.6.1 Seeking proportions

If we are to create poly-dimensional figures from basic geometrical forms, we could at first choose the simple proportions of halving or doubling, or, alternatively, splitting into three or trebling. Splitting or multiplying the square or the triangle, we should take the sides as a point of departure. Halving the sides divides the area of the square into four equal parts, which in turn can also be divided in a similar way, and this leads to the series of 1:4, 1:16, 1:64, 1:256, etc. Splitting the sides into three will divide the area of the square into nine equal parts, and repeating this action will lead to the series of 1:9, 1:81, 1:729, etc. Interestingly, the area division of circles will show the same proportions as that of the square and the triangle, though here we have to choose the radius, and not the circumference, as the basis for scale change. Taking the area formula of the circle ($r^2 \times \pi$), we will find that halving the radius leads to an area one-fourth of the original, whereas dividing the radius by three results in an area decreased to one-ninth.

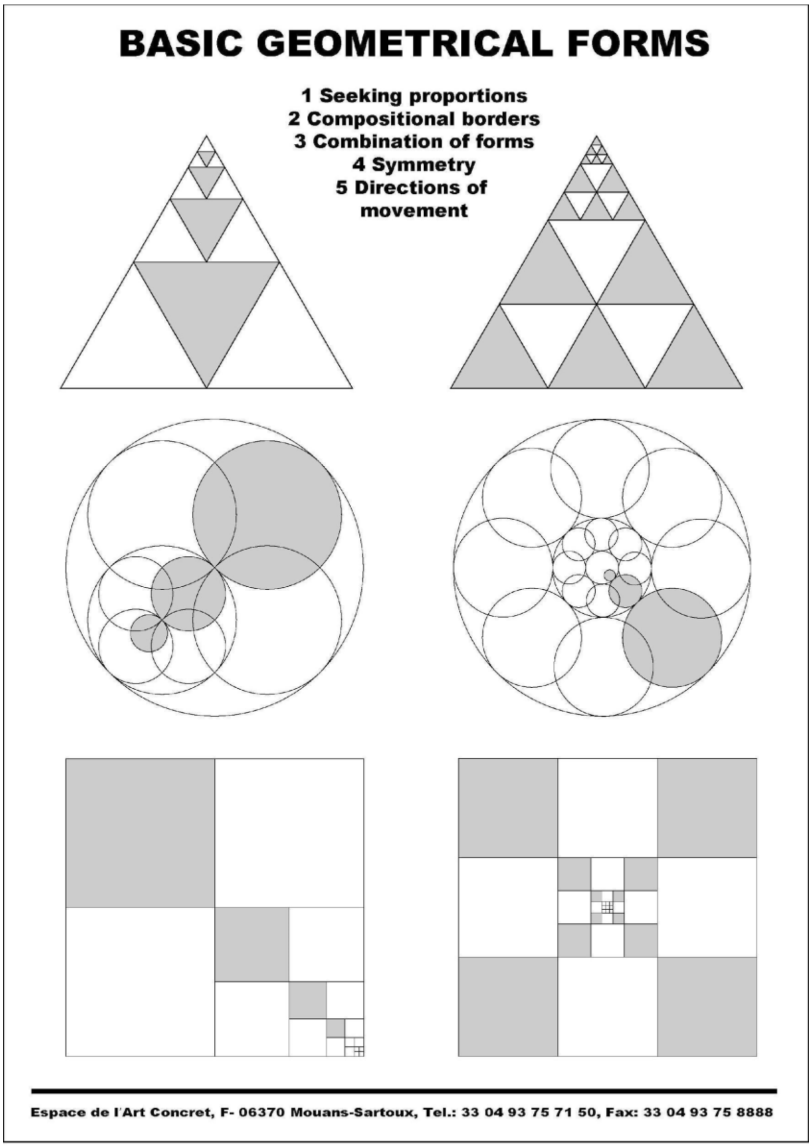


Figure 6: Saxon — To create poly-dimensional figures from basic geometrical forms, at the simple proportions of halving or doubling, and splitting into three or trebling.

2.6.2 Compositional borders

We can choose greater proportions as well, but even halving the forms more than three or four times might make it impossible to use the derived elements for composition, as their areas will become too small (1:256 of the original, for instance). They may even vanish from our sight. In the opposite case, increasing the areas more than three or four times will lead to forms not only unsuitable as compositional elements, but also too huge to store in our home, or even in gallery spaces. Just imagine an exhibition opening where the visitors are looking for the works of art on the walls, while the edges of those works might reach as far as the border of the town.

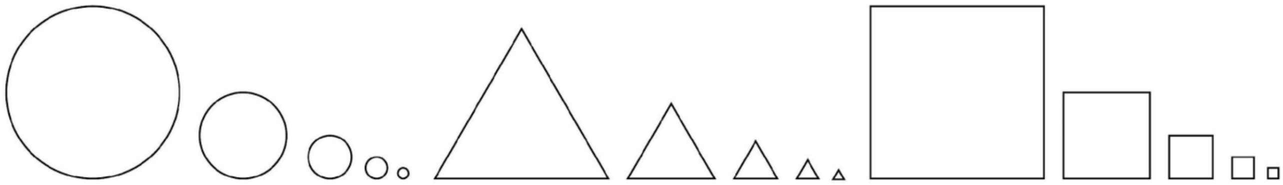


Figure 7: Scale-shifting symmetry
Increasingly smaller elements are obtained by halving the sides of the basic shapes and the diagonal of the circle

Nevertheless, we need not despair: in their physical realization, three or four such manageable scale changes prove more than sufficient to incite in our minds the poly-dimensional movement leading to an infinity of similar leaps.

2.6.3 Points of connection and directions of movement

As the next move, we must find the points of connection between forms. In the case of angular shapes, these will be the corners and random points on their sides. In the case of circles, we can choose any point on their circumference, but if we take a semicircle, a sector or an arc segment, even the diameter can be used as point (or line) of connection.

In all three cases, the geometrical centres of planes can also serve as points of connection, but we can even choose randomly any other point of the surface. What matters is that poly-dimensional structures emerge only if the forms of different levels are connected in a logical order, following a unified rule.

While constructing images, we can differentiate between two directions of movement, as well as their combination:

A / INTERIOR: Using the smaller forms and different tones, we move inwards into the larger form. Thus, we remove parts of it; we create a hiatus, and proceed towards deconstructing its area.

B / EXTERIOR: Following the points of connection and the sides we move outwards from the larger forms, and by adding the smaller forms, we increase the area of the composition.

A+B / MIXED: We follow both directions of construction simultaneously.

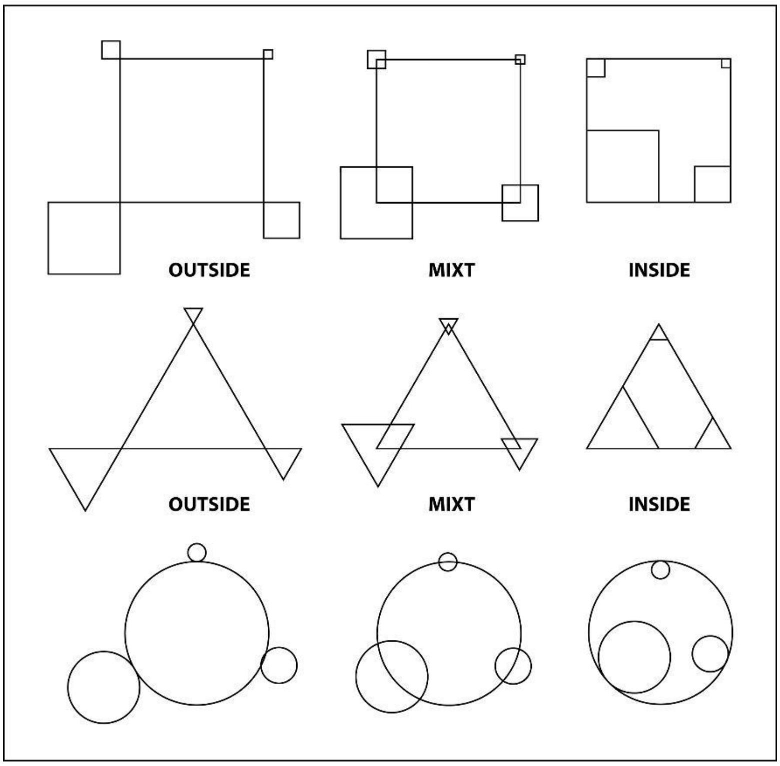


Figure 8: Saxon — The main connection diagrams of the basic shapes, as possible geometric constructions

2.6.4 Combination of forms

Before deciding what tone or color our composition should be, we can combine or merge the forms which have become adjacent during the construction. Moreover, if we connect the free corners, we may get such surfaces (so-called supplementary planes) which are closely related to the basic forms as compositional elements, but also differ from them. After several scale shifts we can naturally facilitate the emergence of a piece of art through the combination of forms and the application of supplementary planes, but we must bear in mind that these procedures will simultaneously decrease the formal openness of the construction. (See examples Figure 9)

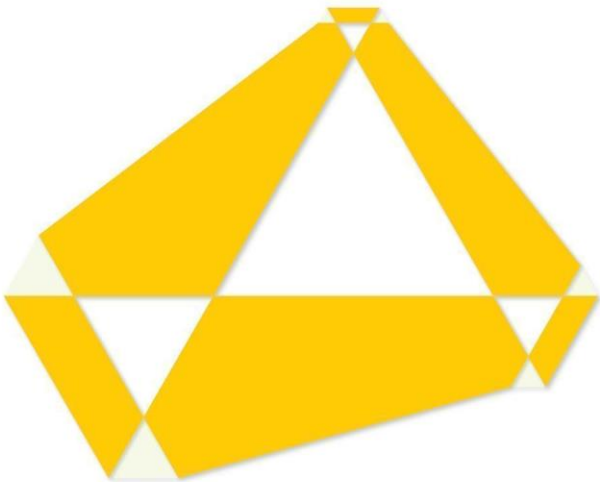


Figure 9: Saxon — Yes-No 1997, oil, wooden board, 45x70 cm
Saxon — Polydimensional triangular spaces I, 2008, oil, wooden board, 150x130 cm

2.6.5 Symmetry

The circle, the equilateral triangle and the square are inherently symmetrical forms (the triangle has an axis of symmetry, while the circle and the square have both axes and centre of symmetry. If we connect the basic forms on various scales, or rotate their combinations according to similar parameters, a system of proportional symmetry will emerge. Given that the parameters of forms and form combinations are reflected within themselves, symmetry does not generally favor the openness of geometric constructions. We can resolve such effects of symmetry through the merging of forms, the use of supplementary planes and the logical reorganization of the elements. (See Figures 10, 11)

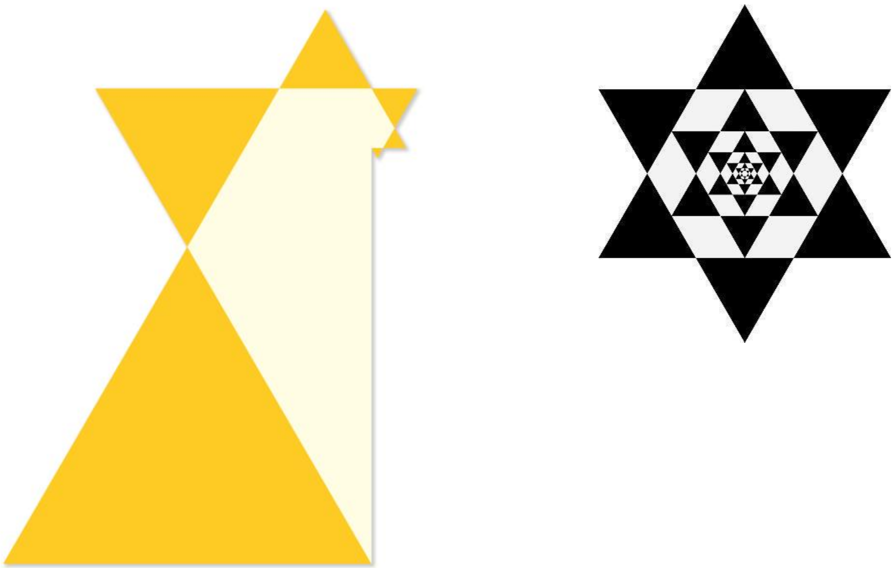


Figure 10: Saxon — Star Poly-D 2004–2008, acrylic, laminated canvas, 150x200 cm, and a diagram of events

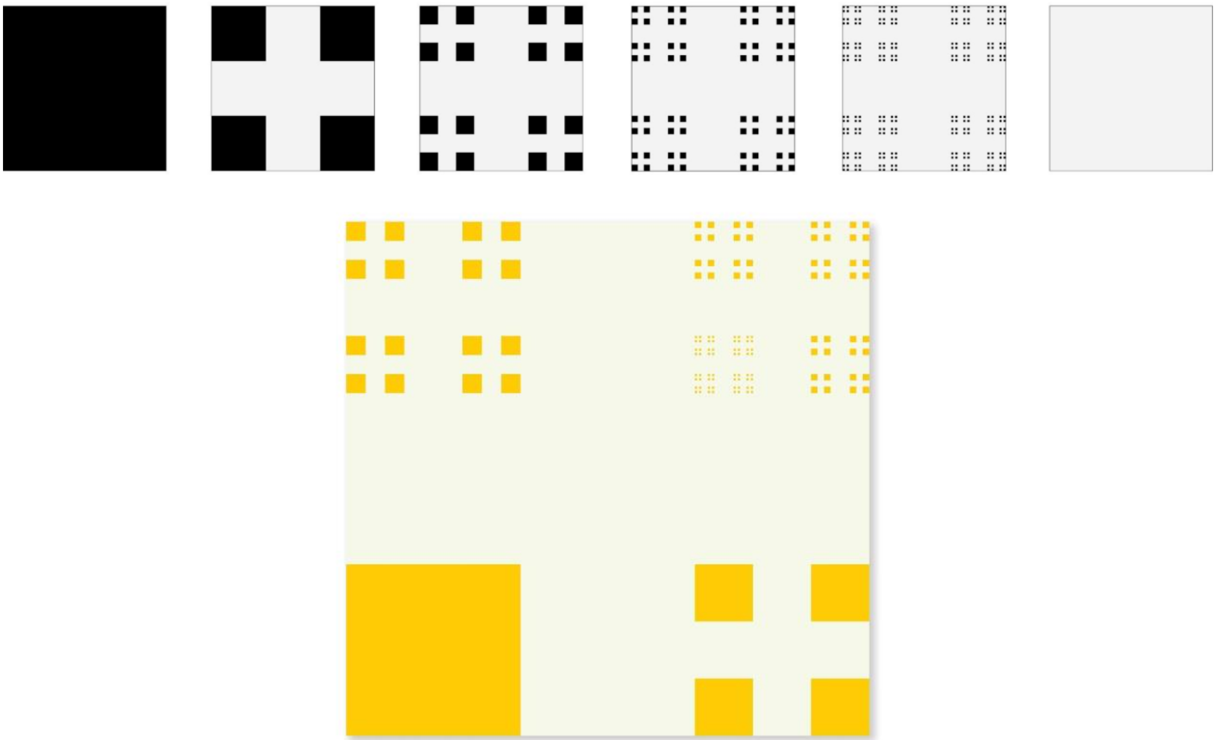


Figure 11: Saxon — Intangible passage 1997, oil, wooden board, 152x152 cm, and a diagram of events

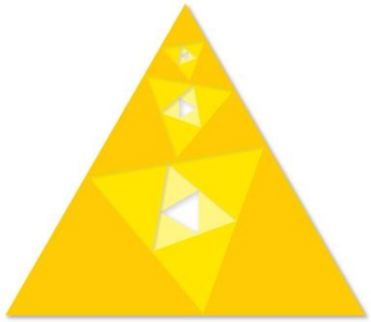
2.6.6 Analysis of poly-dimensional artworks

PURE FORMS:

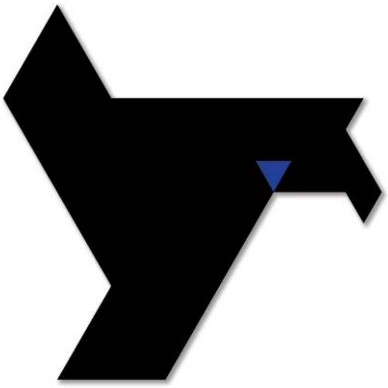
- Square+square
- Triangle+triangle
- Circle+circle

MIXED FORMS:

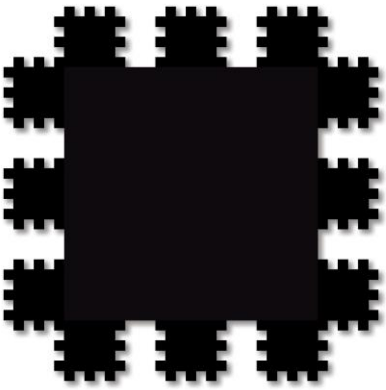
- Square+circle
- Triangle+circle
- Square+triangle



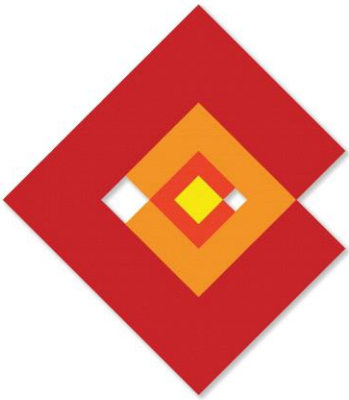
Pure form: triangle+triangle
Proportions of area: 1:4, 1:16, 1:64…
Compositional boundary: six moves
Points of connection: centres
Symmetry: proportional
Direction of movement: interior
Combination of forms; Rotation;
Multiplanal construction; Spaces



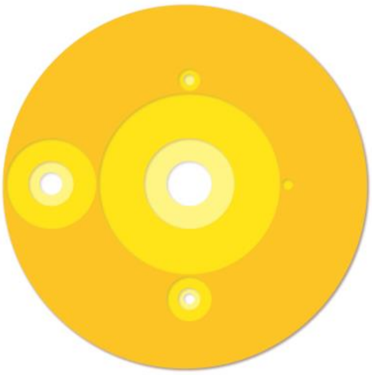
Pure form: triangle+triangle
Proportions of area: 1:9, 1:81…
Compositional boundary: three moves
Points of connection: sides and corners
Symmetry: proportional
Direction of movement: exterior
Combination of forms



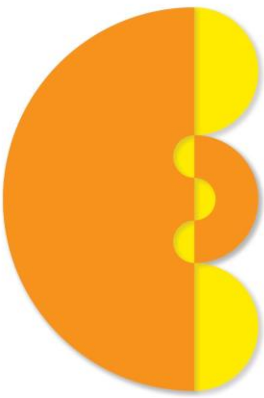
Pure form: square+square
Proportions of area: 1:25, 1:625…
Compositional boundary: three moves
Points of connection: sides and corners
Symmetry: central, axial, proportional
Direction of movement: exterior
Combination of forms



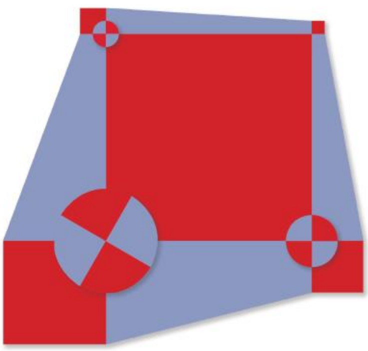
Pure form: square+square
Proportions of area: 1:4, 1:16, 1:64…
Compositional boundary: five moves
Points of connection: centres, corners
Symmetry: axial, proportional
Direction of movement: interior
Spaces



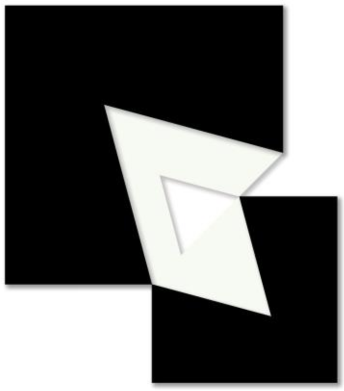
Pure form: circle+circle
Proportions of area: 1:4, 1:1, 1:64…
Compositional boundary: six moves
Points of connection: diameter, centres
Symmetry: central, proportional
Direction of movement: enterior
Combination of forms;
Multiplanal construction;
Spaces



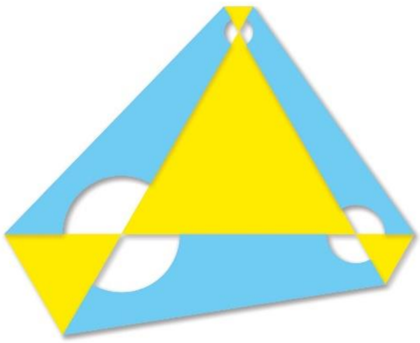
Pure form: semicircle+semicircle
Proportions of area: 1:9, 1:81,
Compositional boundary: three moves
Points of connection: diameter and corners
Symmetry: axial, proportional
Direction of movement: mixed
Multiplanal construction



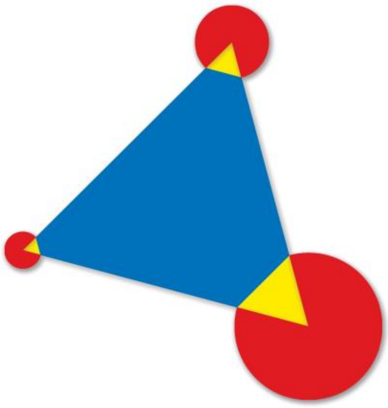
Mixed form: square+circle
Proportions of area: 1:4, 1:16, 1:64, 1:256…
Compositional boundary: five moves
Points of connection: corners, centres
Symmetry: proportional
Direction of movement: exterior
Supplementary planes;
Multiplanal construction;
Mobility



Mixed form: square+triangle
Proportions of area: 1:2,25, 1:5,0625…
Compositional boundary: five moves
Points of connection: sides and corners
Symmetry: proportional
Direction of movement: mixed
Combination of forms;
Multiplanal construction;
Space



Mixed form: triangle+circle
Proportions of area: 1:4, 1:16, 1:64…
Compositional boundary: four moves
Points of connection: corners, centres
Symmetry: proportional
Direction of movement: exterior
Supplementary planes;
Multiplanal contruction;
Spaces



Mixed form: triangle+circle
Proportions of area: 1:4, 1:16, 1:64…
Compositional boundary: five moves
Points of connection: corners and sides, centres
Symmetry: proportional
Direction of movement: mixed
Multiplanal construction

2.6.7 From playful images to Modules of Poly-Universe

While creating the ‘dimension vehicles’ of our journey, we intuitively follow the laws of the basic forms. In this way, polygonal geometric MADI constructions emerge, which are formally free. They are poly-dimensional fields pulsating between micro- and macrocosmic, real pieces of creative art. Actually, the image that has such a spiritual content will construct itself, provided that we approach it with appropriate sensitivity and humility. Now comes the joy of discovery, the inventive solution of the problem. We can consider the problem solved once all our questions have been answered satisfactorily during the visual dialogue. This is the point where the work of art in its physical reality becomes complete.

During the play of forms, however, the squares, triangles and circles may lead to compositions which do not yet fulfill the criteria of a piece of art. But if we take two of these elements, and attach them by various points of connection, we will have at least a dozen of 'playful images.' These images will remain open in the physical sense of the word, too, as we can continue the construction using more compositional elements. Infinitely complex strings of images will thus unfold before our eyes.

Saxon thus arrived at the creation of the elements of the Poly-Universe, while innovating in the structure of the image by using the possibilities of combining primary colors.

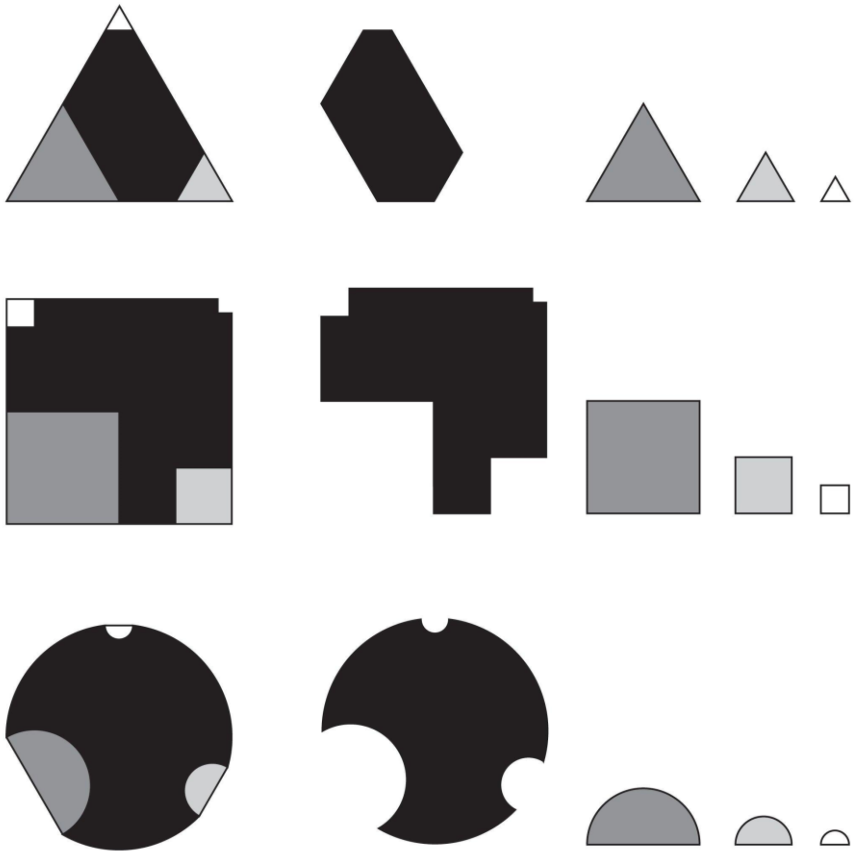


Figure 12: Saxon — Poly-Universe game basic elements and its proportions (triangle, square, and circle)

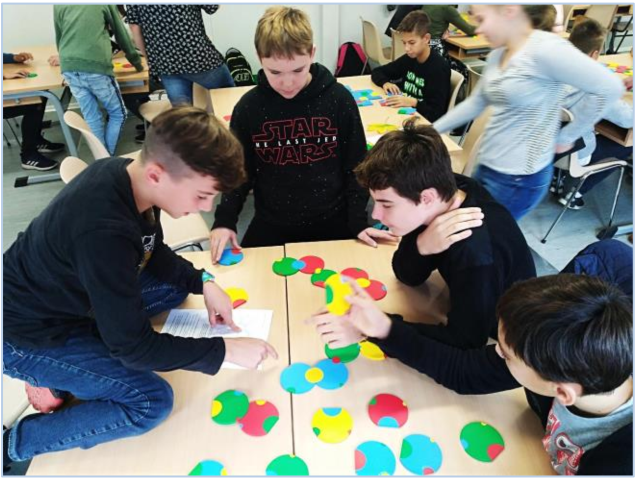


Figure 13: Poly-Universe tool workshops with students in Primary Schools, Hungary

EPILOGUE

'What János Szász Saxon means by poly-dimensionality is approximately covered by that interdisciplinary concept which the sciences using mathematical proportions and their visual representation have termed fractal geometry. Fractal geometry is interested in the objects revealing a peculiar version of symmetry. Every type of symmetry results in a kind of invariable behavior (axial symmetry, for example, shows equal extensions as regards the original forms and their mirror images). Fractal forms are invariable as far as changes of scale or dimension are concerned. In other words, after random scale alterations, the characteristic details of the original form will eventually recur. This mode of invariable structures only became a key question of different sciences in the 1970s, but since then it has led to a radical paradigm shift in various fields of the natural and social sciences. However, the complexity of these problems, and their heavy reliance on mathematics have hindered the representatives of the creative arts to go beyond the surface features of this discipline, despite that proportions have since ancient times been the fundamental elements of music and art.

Remarkably, János Szász Saxon never uses the terminology of fractal geometry while describing his efforts. This, in my opinion, can be explained with the simple and (to me) imposing fact that it was not the recent hype about fractal geometry which made Saxon interested in the forms invariable to shifts of scale; he discovered these compositions completely individually, departing from the achievements of Constructivist art. This in itself could already be an international sensation. Saxon, at the same time, is an excellent creative artist, who has managed to build up his own poly-dimensional art based on very simple forms, resulting in a very persuasive visual language. To a person like myself, who, living in West Europe, has enormous literature on similar visual experiments deriving from mathematical axioms, well, to me, Saxon Szász is so to speak the only convincing evidence that this theory of proportions, rooted basically in scientific theories, can also be realised in creative art. To our even greater satisfaction, János Szász Saxon can think and put his thoughts into words as well, and he has found a way to retain the conceptual and rhetorical background of Constructivism while writing about the artistic representation of forms invariable to shifts of scale.'

(Géza Perneczky art historian, Cologne — SAXON: Dimension crayon, published by Espace de l'Art Concrete 2001)

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2.7 Friezes, rosettes and Poly-Universe

2.7.1 Symmetry, plane isometries

Though symmetry is a wider concept, by symmetry people mostly think of reflection. Generally, we consider an object symmetric if the given object is invariant under an operation (that is, some properties are preserved) (Darvas, 1999). Dissymmetry is a breaking of symmetry: the phenomenon, law or shape keeps the symmetry in its main concept, but not necessarily fully symmetric in its details. We can meet dissymmetry quite often when observing nature and art. Consider the human face or a panel of the coffered ceiling in a church. Asymmetry simply means the lack of symmetry. In this study, we cover plane isometries and frieze and rosette groups generated by them. The classification of the four plane isometries by orientation preserving property and the existence of fixed points can be seen in Table 1.

	There is a fixed point	No fixed point
Preserves orientation (motion)	Rotation	Translation
Reverses orientation	Reflection	Glide reflection

Table 1: Classification of plane isometries

2.7.2 Symmetries in the PUSE methodology book

When studying symmetries in this essay we imply forms but ignore the colors. More precisely, if we say that, for instance, two figures can be mapped onto each other by translation, then after translation, the same forms but not necessarily the same colors will cover each other.

In section A of the geometry chapter of the book some tasks directly ask about isometry. In 110_A the task is to color the reflected image of triangle, square, circle elements; in 102_A they build shapes that remind them of rotation (Figure 1). It is well worth observing the ratio of symmetric and non-symmetric shapes created by children during tasks allowing for free play and the flow of creation.

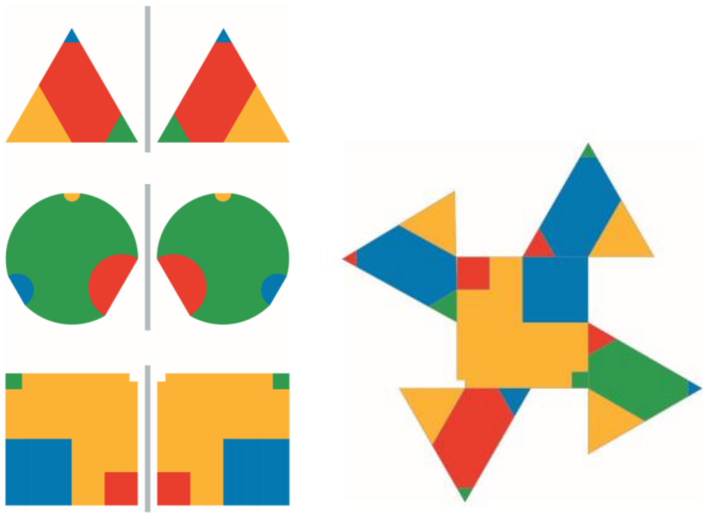

Figure 1: Figures for task sheets 110_A and 102_A.

Task 505_A is the following: 'Work in pairs: both of you should construct a shape of 3-4 elements that your partner cannot see. Then show it to each other for a few seconds and cover them. After that draw the shape your partner constructed for you from memory. Check each other's solution if you are ready. If you succeed, continue the game by constructing the given shapes using the elements of your own sets.'

It is useful to ask about the symmetries of the constructed shapes. The teacher methodology sheets provide some of the most interesting symmetry constructions (Figure 2).

Apart from the top right picture, each construction seems to have reflection symmetry. Having a closer look at them, though, we notice that only the top left construction has perfect reflection symmetry, the other two just almost. What is the reason for this imperfect symmetry? The axis of symmetry is not where two shapes meet (as with the top left figure) but it crosses the figure – and none of the Poly-Universe elements is mirror symmetric.

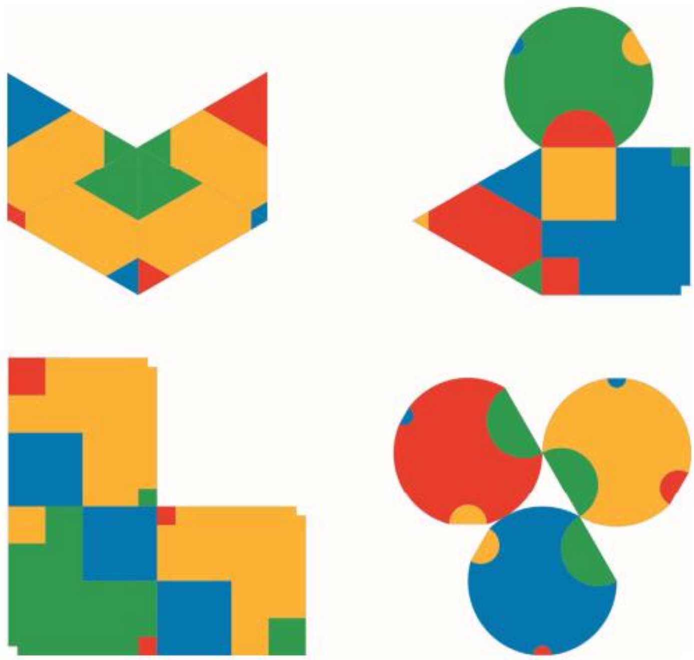

Figure 2: Figures for task sheet 505_AB

2.7.3 Generally about frieze symmetries

The isometry groups containing a single translation are *frieze groups*.

There are exactly seven ways of creating (infinite) linear patterns (friezes) which are generated by the (infinite) repetition of one single motif. The different codes and patterns of these subgroups (represented by letters) are shown in Table 2. The explanation for codes is the following:

Letter *p* at position 1 is the notation of the initial pattern (comes from the word pattern).

At position 2 there is a letter *m* if the pattern contains reflection symmetry with an axis perpendicular to the direction of translation. Otherwise, we write *1*.

At position 3 there are letters *m* or *a* if the pattern contains reflection symmetry with an axis parallel to the direction of translation, or a glide reflection, respectively. Otherwise, we write *1*.

At position 4 comes the order of rotation. It can be proven that frieze patterns can only have rotation centres of order two. Otherwise, we write *1*.

Group code	Pattern
p111	...LLL...
p112	...NNN…
p1m1	...DDD...
p1a1	...bpbpbp…
pm11	...AAA...
pmm2	...HHH...
pma2	...ΛVΛ...

Table 2: Classification of frieze symmetries

2.7.4 Friezes and the Poly-Universe

As we can see on Figures 2–16, Poly-Universe constructions allow for all seven frieze symmetries. Those patterns which have reflection symmetry with an axis parallel to the direction of translation can only be assembled by arranging the elements at least into two lines (because Poly-Universe elements do not have inner isometries).

Figure 3: p111

Figure 4: p112

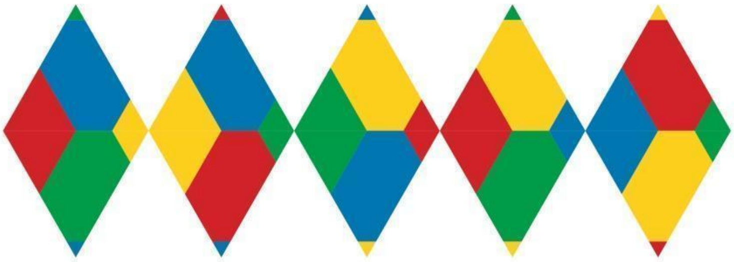

Figure 5: p1m1

Figure 6: p1a1

Figure 7: pm11

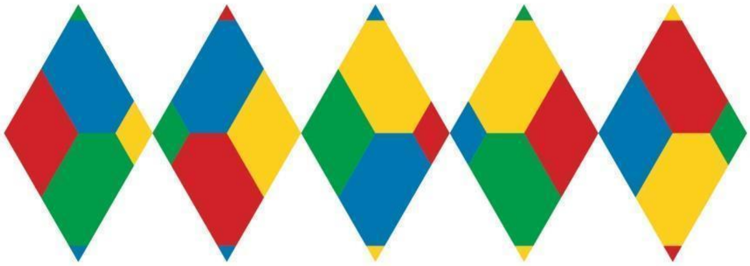

Figure 8: pmm2

Figure 9: pma2

Now let's see which tasks of the book can be extended by questions on frieze symmetry.

Task 104_A asks to build quadrilaterals by assembling triangle elements. We can establish that an odd number of triangles make up trapezia, with an even number we get parallelograms. This task can be extended by asking the following question: with connections of the same color and size, what type of symmetry does the constructed linear pattern have? The answer: frieze pattern with p1a1 symmetry pattern (Figure 6).

This extension may be applied to other tasks, e.g. 106_A, 130_BC. At 306_A, the task is to identify the rules of the connected triangle elements. Apart from the many rules, it is worth adding symmetry rules like p111 and pm11 frieze patterns.

Figure 10: p111

Figure 11: p112

Figure 12: p1m1

Figure 13: p1a1

Figure 14: pm11

Figure 15: pmm2

Figure 16: pma2

Finally, the most beautiful frieze pattern from Poly-Universe elements in the book is the decorative frieze at the beginning of chapters, constructed by János Szász Saxon. This p1a1 frieze, in Figure 17, contains glide reflection.

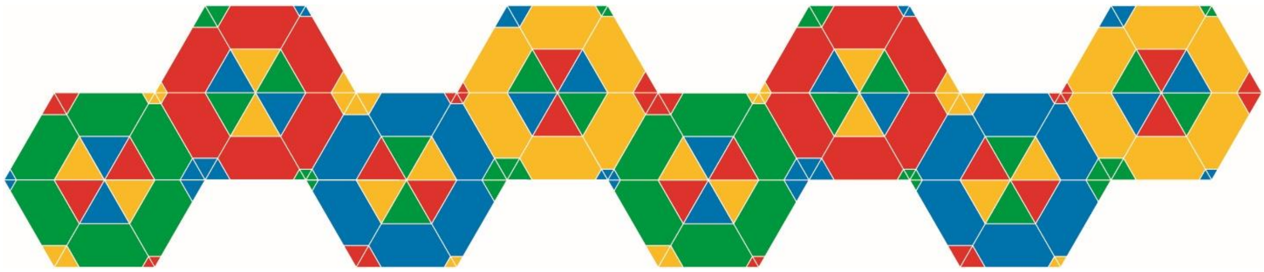

Figure 17: p1a1 frieze, decorating the PUSE book.

2.7.5 Generally about rosettes

The Rosette (rose window) group is a translation-free group, its name comes from church windows. There are infinitely many rosette groups which can be classified into two substantially different subgroups. One of them consists of groups of rotations about a single point: rotations with integer multiples of $2\pi/n$. They belong to the cyclic rosette group, written C_n. These groups do not have reflection symmetry. Besides the mentioned rotations, if the group also have n reflection symmetries with axes going through the centre of rotation, they belong to the dihedral group, written D_{2n}.

Figure 18: Left: St Peter and Paul church Gorlitz, C_6 rosette group. Right: Cambridge, Cambridgeshire, England, UK, D_{10} rosette group. (photo by Leo Reynolds)

Rosettes are not simply on buildings, we may find plenty of them in nature: when we cut an apple or orange in half; or when we marvel at a lovely flower or cactus, and we may even find starfish from the D_{10} symmetry group. Among old door locks and manhole covers, we can discover rosette symmetry; what's more, looking at our cars' hubcaps we can quickly find their 'rosette code', and we could continue the discovery in further areas of everyday life.

2.7.6 Rosettes and the Poly-Universe

The book contains many examples of rosettes assembled by triangle and square elements. We can arrange 6 triangles to make up a hexagon with D_6 rosette symmetry. Examples of this can be seen on task sheets 125_B and 212_B (Figure 19).

Using the entire set, large hexagons assembled by triangle connections of the same color and size have D_6 symmetry, see Tasks 130_BC and 212_B. The picture on the right in Figure 19 is an example of this. Task 212_B from the Combinatorics chapter asks for the number of constructions with the same size and color connections. There are 12 ways to construct smaller (of 6 triangles) and larger (of 24 triangles) hexagons as well with the given conditions. We can establish that each construction has D_6 rosette symmetry.

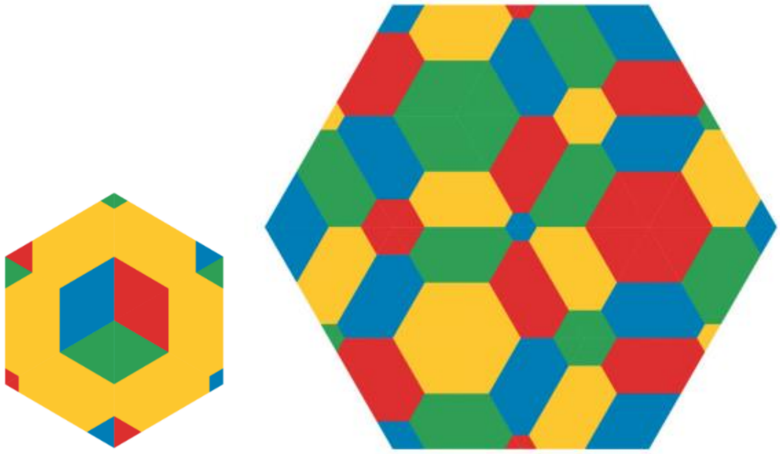

Figure 19: Rosettes from triangles on PUSE task sheets.

Flipping through the book we notice that the shapes we construct from triangles can either have no symmetries or have dihedral symmetry (see Figure 19). Would it be possible to construct rosettes with cyclic symmetry, using triangles? The answer is yes, but to keep the restriction of total side connection, we can only work with connections of the same color and give up on same size connections.

See Figure 20 for examples, all three of them are C_3 cyclic rosette groups.

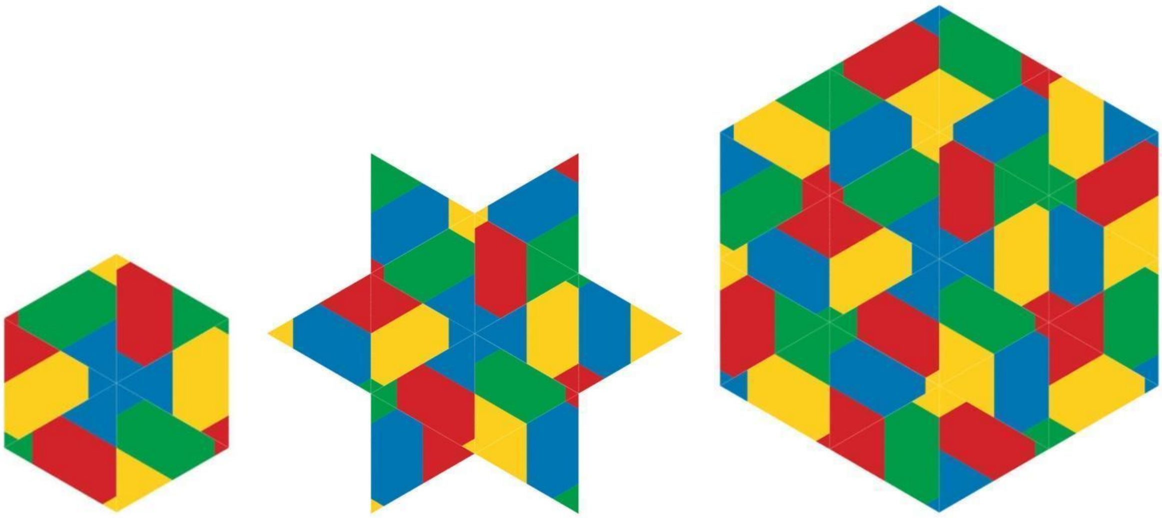

Figure 20: Cyclic rosette groups from triangle elements.

Nice rosettes can be constructed from square elements as well. First, let's examine the task sheets from this perspective. Many questions could be raized about symmetry in connection with Task 108_A; now the given solutions do not contain any symmetries at all. One possible solution of 231_C on the teacher sheet has D_4 dihedral symmetry (Figure 21).

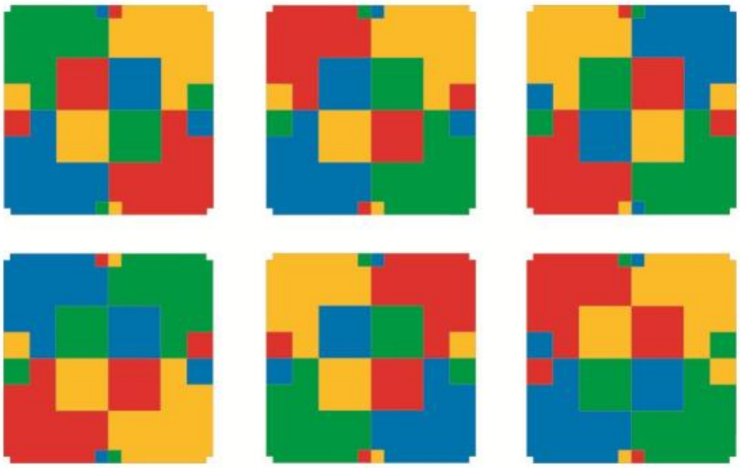

Figure 21: Rosettes from square elements on PUSE task sheets.

In the case of squares also, we can ask whether it is possible to construct a cyclic group. The answer is yes, two examples from Task 203_A represent the C_2 cyclic group (see Figure 22).

Figure 22: Cyclic rosette groups from square elements

We can also ask whether it is possible to construct bigger shapes (using more than four squares) of rosette symmetries.

Finally, on rosettes, we need to mention Task 508_AB which encourages group work constructions, using all three sets. For this, one solution on the teacher sheet is also a D_6 dihedral symmetry rosette (Figure 23).

As we can see, Poly-Universe offers endless possibilities, and there are almost unlimited questions to raise. When we listed friezes and rosettes that can be constructed from Poly-Universe, we did not aim to be exhaustive. We did not give an answer to the question asking for the number of ways to assemble each pattern; we simply wanted to show that there are suitable ways of combining the Poly-Universe triangle and square elements so that all frieze and rosette patterns can be constructed. We would like to encourage the Reader to look for further nice patterns and try to answer the proposed questions for the Poly-Universe circle elements, and for mixed sets. Hopefully, we will return to these questions in a future study.

Figure 23: Rosette from mixed Poly-Universe elements.

2.8 Symmetries in Portuguese and Hungarian folk art, and influence to Poly-Universe

2.8.1 Symmetries in Portuguese tiles

Tile has a prominent place in Portugal, being a constant presence in interior or exterior coatings of buildings. All heritage is a cultural heritage that defines the memory of a people and that is projected into the future: Portuguese tiles are a good example of this identity. The 'Pombalina' era was another period of great importance in Portuguese tiles. After the Lisbon earthquake in 1755, the tile gained a remarkable utilitarian function, despite the formal character of the tiles, the decoration was not neglected, which revealed itself in very characteristic floral patterns. In the study of the symmetry of figures in the plane, the four possible types of symmetry are considered: reflection symmetry, translation symmetry, rotation symmetry and sliding reflection symmetry. A more complete study of the symmetries of a figure involves considering the possible (discrete) groups of symmetry of plane figures: rosettes, friezes and patterns. The symmetry groups of rosettes are of only two types: cyclic, Cn, or dihedral, Dn. The plane symmetry groups are of seven types: p111, p112, p1a1, pm11, p1m1, pma2, pmm2 (Martin, 1982). These types of symmetries will be found on tiles existing in the cities of Aveiro and Ovar.

Aveiro Tiles ('Azulejos' de Aveiro)

The use of tile in the architectural facades of the city Aveiro begins in 1857, the year in which the first building whose facade appears it is entirely covered with tiles, at the beginning of the 20th century the Art Nouveau movement reached Portugal and found in tiles a privileged support for its expression in Aveiro. Along with the beginning of production of outdoor tiles, 'Fábrica da Fonte Nova' was born in 1882, a ceramics industry.

Figure 1: Frieze type p112

Figure 2: Frieze type pmm2

Figure 3: Frieze type p1a1

Figure 4: Rosette type C2

Ovar Tiles ('Azulejos' de Ovar)

Ovar's tiles are a heritage and a differentiating element that marks the city's image and history. Ovar has the current name of tile museum city because it presents a representative set of tiled facades, dating from the 19th/20th century, which create a unique atmosphere.

Figure 5: Frieze type p111

Figure 6: Frieze type p1a1

Figure 7: Frieze type D_4

2.8.2 Frieze symmetries on Hungarian cross-stitches

Each frieze symmetry pattern also appears in the Hungarian decorative art from the era of the Hungarian conquest of the Carpathian basin (Bérczi, S. 1986), but similarly can be found in other nations' art. We can learn about it in *Eurasian art* collection edited by Szaniszló Bérczi, which can be downloaded from this website: *http://www.federatio.org/tkte.html.* We come across friezes everywhere: when walking along an old street if we look up to the ornamentation of the buildings – or look down, noticing the border of an ornate floor or panelling. Each of the seven frieze patterns appears on Hungarian cross-stitch embroidery as we read the study of István Hargittai and Györgyi Lengyel. (Hargittai I. & Lengyel, Gy. 2003).

On a website *(http://qtp.hu/xszemes/mn.php)* one can find a huge array of cross-stitch patterns selected from the collection of the Hungarian Museum of Ethnography. A question arose: would it be possible to discover all frieze symmetries among these? The answer is positive, as Figures 3–7 show. Originally these patterns could be found on tablecloths, bedsheets, pillows, shirts, overlay patterns on traditional Hungarian coats. The most common patterns were the ones with only vertical reflection symmetry or the ones with vertical and horizontal reflection symmetries. The most difficult task was to find patterns without inner symmetry, or patterns which have reflection symmetry with a horizontal axis but no vertical axis.

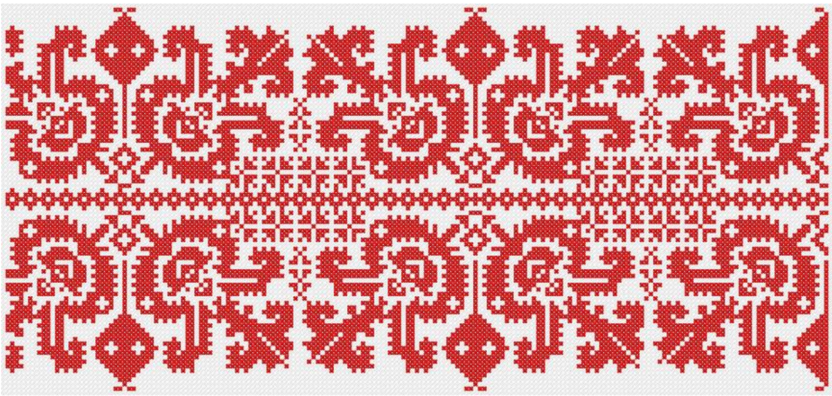

Figure 8: pmm2

NM 8823

Pillow-end pattern, originally with long-armed cross stitch and cross-stitch embroidery
Time of creation: end of 19th century
Place of creation: Diósad

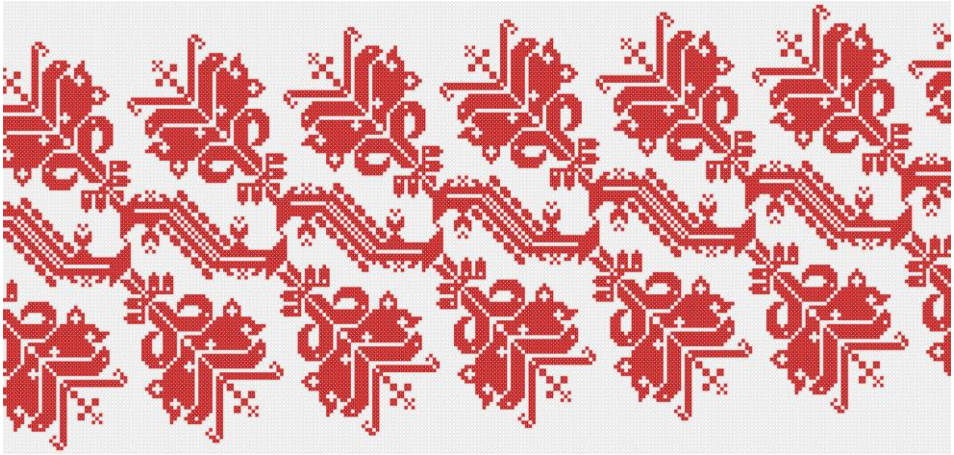

Figure 9: p112

NM 126854
Pillow-end pattern
Time of creation: the second part of the 19th century
Place of creation: Kalotaszeg (Kolozs county)

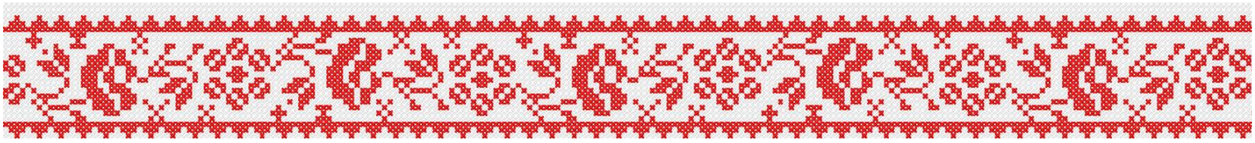

Figure 10: p1a1

NM 127238
Table-cloth. A mix of long-armed cross stitch and cross-stitch.
Time of creation: First quarter of the 20th century
Place of creation: Kalotaszeg (Kolozs county)

Figure 11: p111

NM 126917
Pillow-end
Time of creation: 2nd part of 19th century
Place of creation: Kalotaszeg

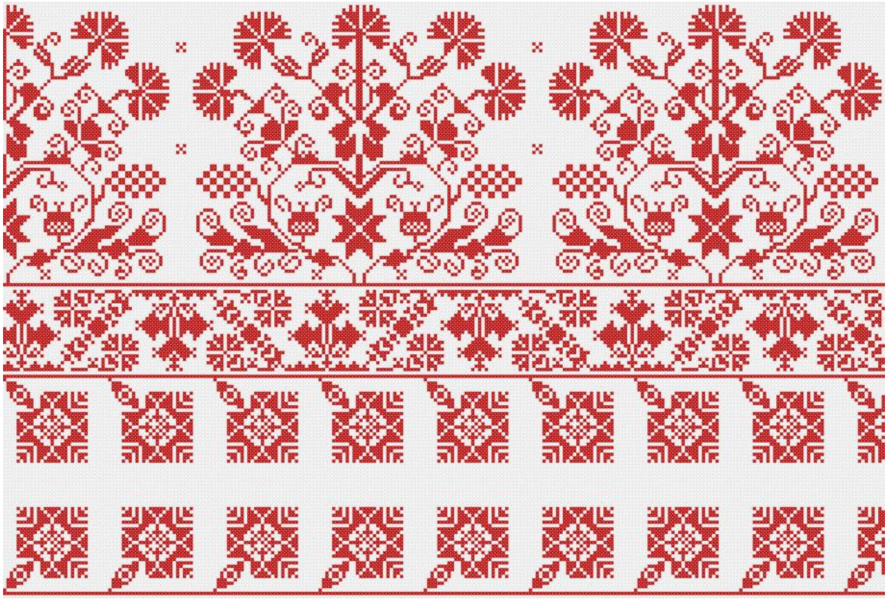

Figure 12: This cross-stitch shows 3 frieze symmetries: the upper pattern has pm11, the middle pma2, and the lower pattern p1m1 symmetry.

NM 51.14.655
Sheet-end, originally cross-stitch and long-armed cross-stitch embroidery.
Time of creation: turn of 18-19th century
Place of creation: Transdanubia

2.8.3 Poly-Universe wallpaper groups with the colors of Portuguese tiles

With Poly-Universe, we can not only lay out friezes and rosettes, but also display the symmetry groups (crystal groups) of the plane, the so-called cyclic surface divisions.

What is cyclic surface division?

We are looking for the answer to the question: how many ways can a plane be covered by congruent shapes without gaps and overlaps, so that certain transformations (shifts, reflections, rotations) bring the whole pattern into itself, while the adjacent domains are superimposed on each other? The answer to this question is 17, because there are 17 crystal groups in the Euclidean plane.

Some of these are presented using the square Poly-Universe tiles, and the coloring follows the colors of traditional Portuguese tiles:

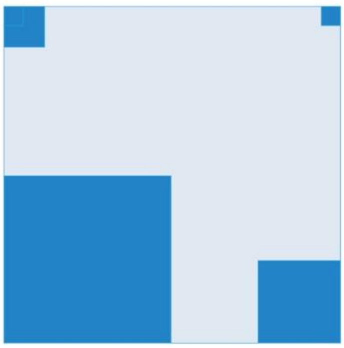

Figure: Base tile

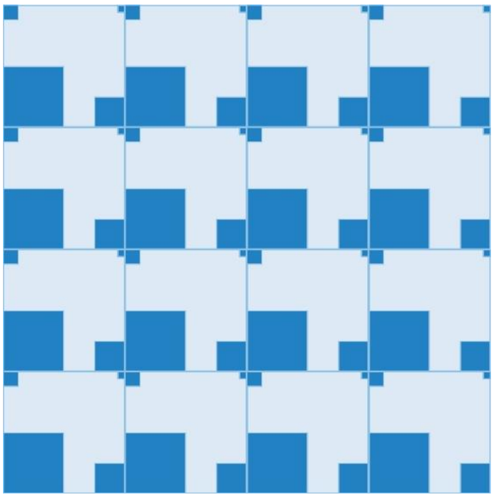

Figure 1: p1

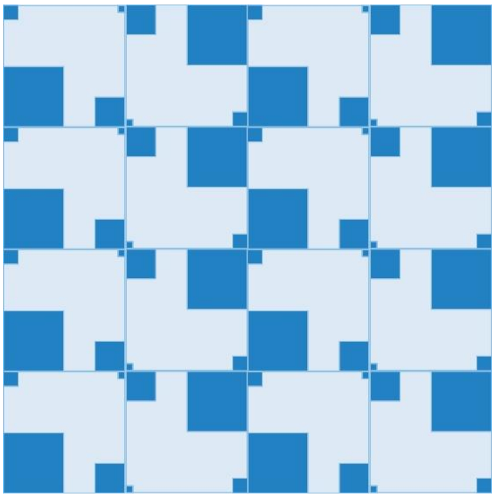

Figure 2: p2

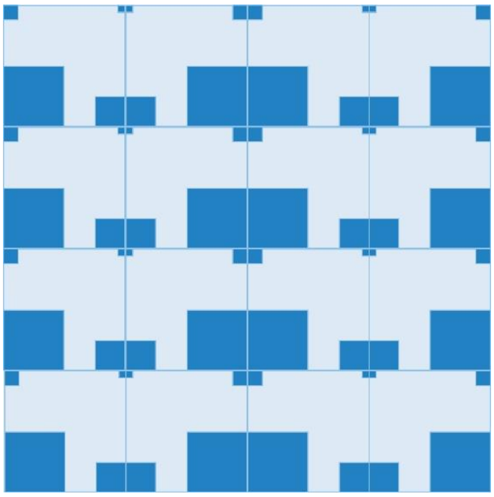

Figure 3: pm

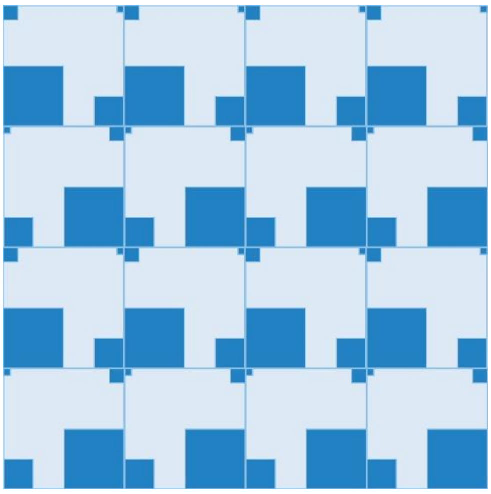

Figure 4: pg

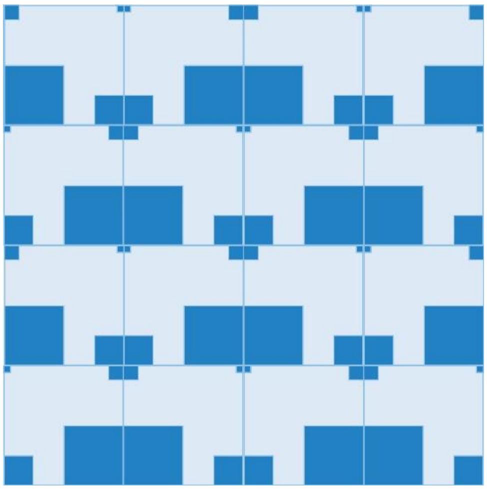

Figure 5: cm

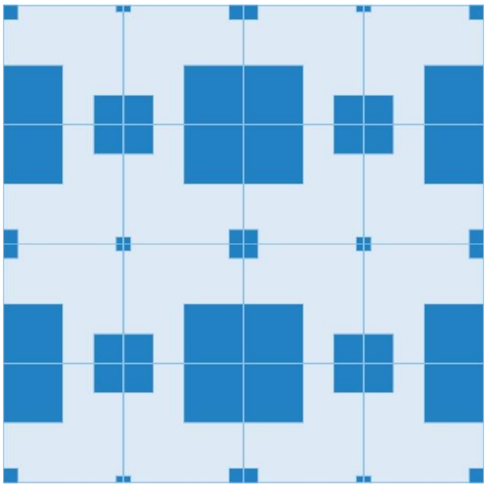

Figure 6: pmm

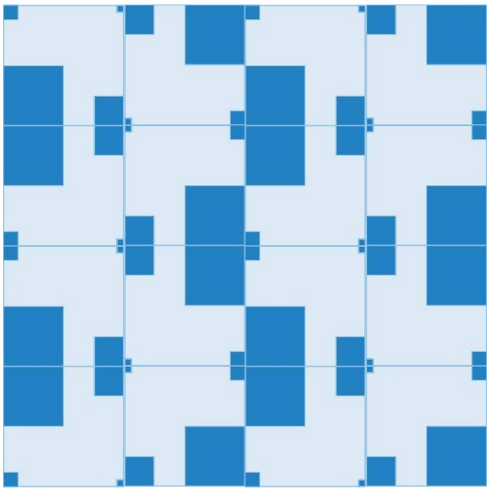

Figure 7: pmg

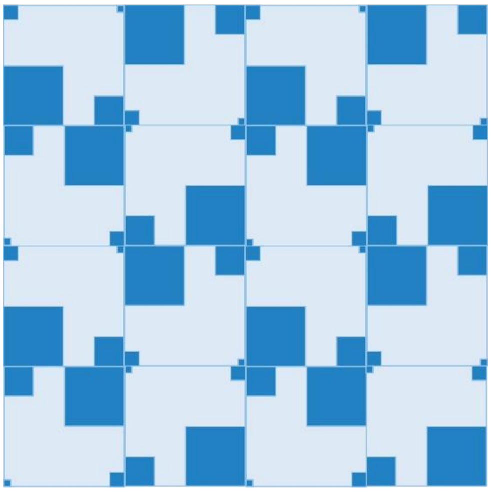

Figure 8: pgg

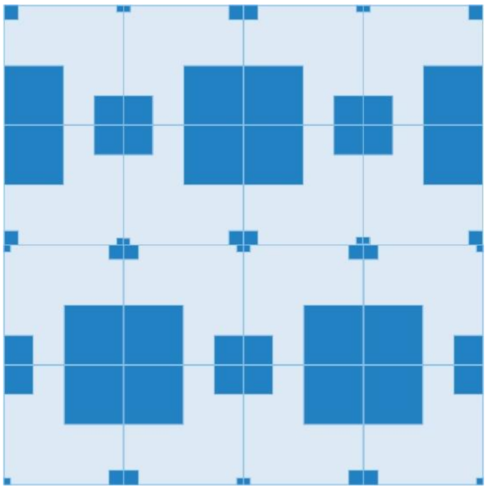

Figure 9: cmm

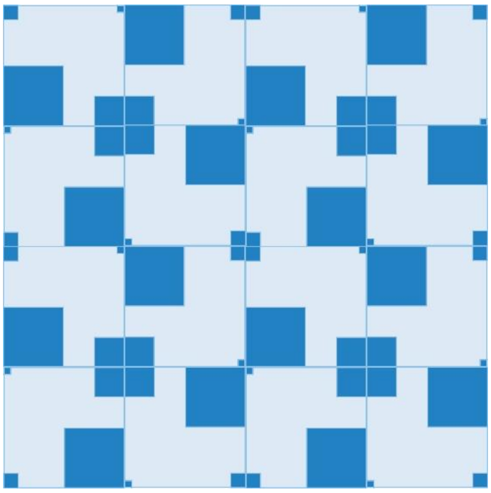

Figure 10: p4

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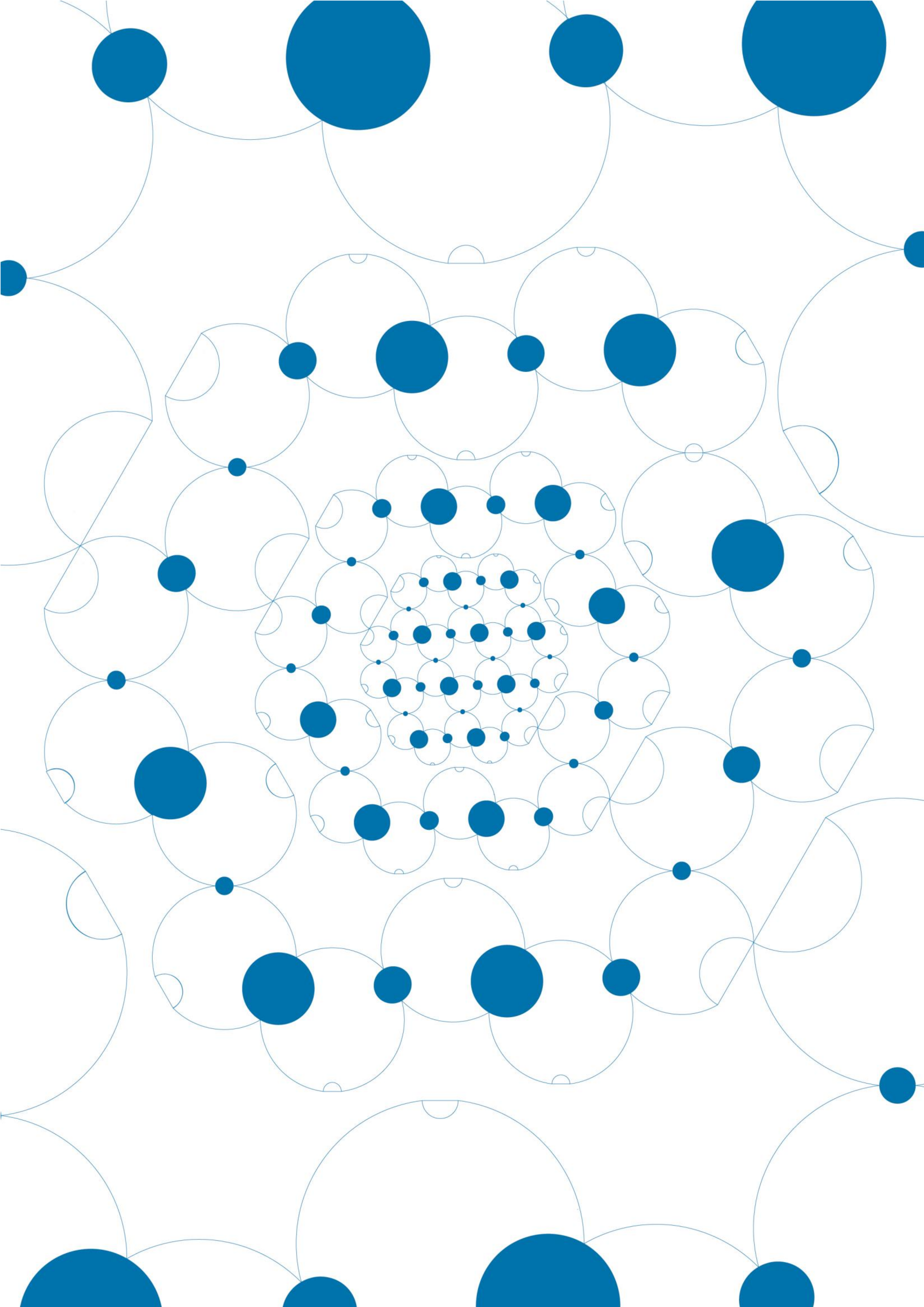

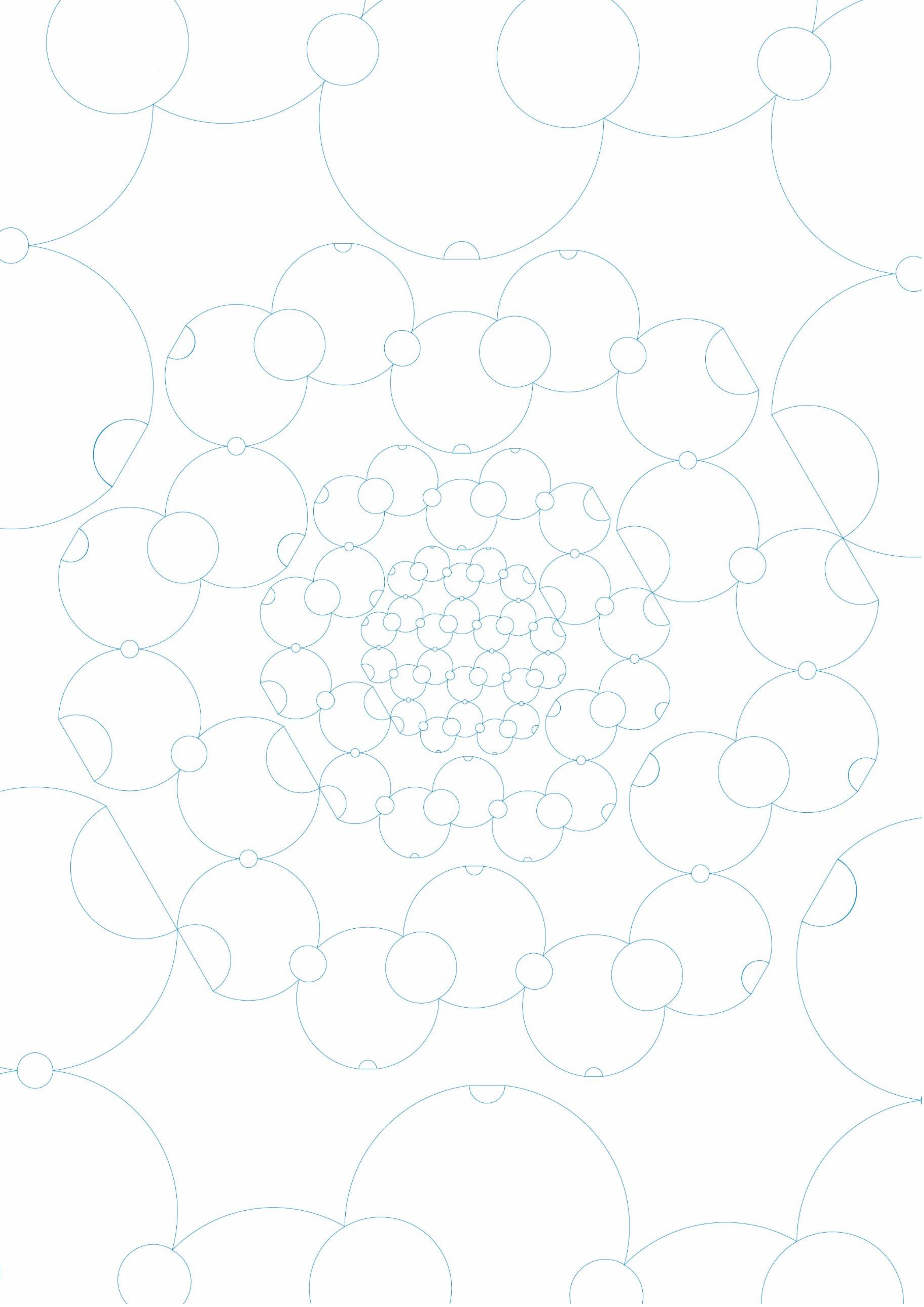

III METHODOLOGICAL BACKGROUND

3.1 Cooperative learning & spontaneous cooperation in learning contexts

Cooperation is currently much advocated, and its importance is recognized at European level in the framework of education for democracy and citizenship, which has been reflected in the educational plan in the definition of Students' Profile by the End of Compulsory Schooling or in OECD2030 Transformative Competencies. It is therefore up to the school to promote strategies that lead to their development, namely through the use of cooperative learning. Cooperative learning has been associated with academic learning and is now advocated as a form of high-impact instruction. In this chapter, we try to analyze the origins and foundations of the movement called cooperative learning, the main characteristics and strategies developed, and the benefits or academic and social outcomes of this kind of methodology, considering also the importance of forms of spontaneous collaboration which can take place in the classroom and which prove to be beneficial.

3.1.1 Foundations & relevance

The origins of this classroom structure are linked to a movement, called cooperative learning, which was developed mainly after the 1960s in the 20th century and originated in the USA, although the advantages of group work were already present in the thinking of prominent European pedagogues of the 19th century, such as Herbart, Froebel, Pestalozzi or Dewey. For example, the latter author, already in 1916, was an advocate of cooperative methodologies, as he understood the school as the 'mirror of social life'; in it, students were to be subjected to methodologies that would promote the development of the true democratic spirit (Arends, 2012). Students, always holders of personal experiences, should take an active role in the school, not only by training, experimentation, but also by training in the exchange of ideas, sharing experiences, dialogue, discussion, and consensus, learning to defend an idea but also to give up their opinion in favor of the team and respecting the other, thus promoting a democratic school environment (Arends, 2012).

As referred to in the previous paragraph, in the mid-1960s, several researchers began a systematic work on developing a body of knowledge that forms the basis of cooperative learning (e.g. David, Roger Jonhson, Slavin, Aronson, Cohen, Kagan, Sharan). Cooperative learning developed more as a result of a practice that was proving successful than as a result of a generally accepted theory, finding its foundations in theories of cognitive and motivational development, as well as in Lewin, Lippit and White's studies of group social climates and Deustch's on the effects of competition and cooperation in groups. The movement in defense of cooperative learning emerged with studies carried out by Lewin, in the 1930s, on group dynamics and its influence on children's interactions when applied in a school context, concluded that school results were better in working groups with a cooperative and democratic spirit than in groups with a more autocratic nature (Gillies & Ashman, 2003). Deustch et al. (1992, 9) research pointed that 'from the perspective of climate, conflict resolution and cooperative learning training (...) seem to be worthwhile'.

Cooperative learning has since then been linked to active and student-centred learning and social interactions/relationships, but also academic learning. In this scope, Slavin (1995, 3) refers 'one of the most important reasons that cooperative learning methods were developed is that educators and social scientists have long known about detrimental effects of competition as it is usually used in the classroom.' The social family capitalizes on our nature as social creatures to further learning and to expand our ability to relate productively to one another.

Cooperative learning is strongly believed to positively affect social development. It is associated with positive interpersonal relationships, self-esteem, critical thinking skills, ability to accept others' perspectives, higher intrinsic motivation, positive attitudes towards subjects, school, teachers and classmates, fewer disciplinary problems as there are more attempts to solve personal conflict problems. Cooperative learning has also been associated with academic learning. In this scope, as referred, cooperative learning is now advocated as a form of high-impact instruction (Knight, 2013), that promotes engagement by assigning each student a task, varying the way students learn; facilitates formative assessment; facilitates differentiated teaching; enables students to build knowledge collaboratively; develops students' communication skills; prepares students for life after school, being also a methodology and classroom structure that promotes inclusion and educational success.

3.1.2 Main characteristics & strategies

Cooperative learning has been considered a teaching or learning methodology included in social models of teaching (Joyce & Weil, 2003). Slavin, in his book, Cooperative Learning (1995, 3), commented: 'Cooperative learning refers to a variety of teaching methods in which students work in small groups to help one another learn academic content. Cooperative learning can then be defined as any systematic and structured learning strategy in which groups of learners work together to achieve a common goal (Knight, 2013, 218) and this is a characteristic that distinguishes cooperation from other forms of interaction. It involves strategies based on principles and procedures, which are different from ordinary group work, constituting an alternative to competitive and individualistic structures, contributing to better cognitive learning and the development of social skills.

Cooperation emerges then as a way to achieve a goal that individually could not be achieved. In fact, one of the components of cooperative learning consists of positive interdependence, which assumes several modalities, namely, the interdependence of purposes, when group members work towards a common purpose, of the task, of resources, the environment/space where the group works. This means that cooperative learning takes place not only when group members work towards a common purpose, but when each member of the group is responsible for the success or failure not only of him/herself but also of the group, which leads students to help their colleagues in order to help themselves (Slavin, 1985). Individual and group responsibility are related to the responsibility of each member to fulfil their part for the group to achieve success. The purpose and responsibility of the group is to strengthen each individual member so that they learn together to do better as individuals.

The learning environment for cooperative learning is characterized by democratic processes and active roles for students in deciding what should be studied and how. The teacher may provide a high degree of structure in forming groups and defining overall procedures, but students are left in control of the minute-to-minute interactions within their groups. This involves establishing rules and schedule for the cooperative work as well and task division within the group. Then, for cooperation to be truly effective, it is necessary to teach students social skills essential to group work: praising, encouraging, asking for help, communicating clearly, accepting differences, listening, helping others, among others, and motivating students to use them. In a cooperatively structured activity, students learn that the teacher is not the only one who can help them, and they develop the academic component along with relationships of solidarity and mutual help (Herreid, 1998).

Arends (2012, 375) referred that six major phases or steps are involved in a cooperative learning lesson, as referred following:

1) 'A lesson begins with the teacher going over the goals of the lesson and getting students motivated to learn.
2) This phase is followed by the presentation of information, often in the form of text rather than lecture.
3) Students are then organized into study teams.
4) In the next step, students, assisted by the teacher, work together to accomplish interdependent tasks. Final phases of a cooperative learning lesson include
5) presentation of the group's end product or testing on what students have learned and
6) recognition of group and individual efforts'.

Based on the general principles mentioned, cooperative learning strategies can have different structures and syntaxes. Following are some examples such as jigsaw, cooperative scripting, Think-Pair-Share, group investigation.

Jigsaw was developed by Elliot Aronson and his colleagues (e.g. Aronson & Patnoe, 1997). Students are assigned to five- or six-member heterogeneous study teams, academic materials are presented, each group divides the content in subtopics and each element of the groups becomes responsible for learning a subtopic. Then new groups are formed that bring together students responsible for analyzing a particular topic. This is called an 'expert group' as it is aimed to analyze in depth the particular subtopic. Following, each 'expert participant' returns to the original group and shares the knowledge created. A quiz is used to test what students have learned.

In Cooperative Scripting, students work in small groups or reciprocal pairs to make a summary of the material that the teacher presented, summarizing the material and sharing it with a colleague.

The Think-Pair-Share, initially developed by Frank Lyman and his colleagues (e.g. Lyman & Foyle, 1990), aims to give students time to think, to respond and to help each other. The teacher can complete a short presentation or students read an assignment or a puzzling situation and consider more fully what has been explained.

Group Investigation students can be involved in planning both the topics for study and the ways to proceed with their investigations. Five- or six member heterogeneous groups are formed, either by the teacher or by the students. Students select topics for study, pursue in-depth investigations of chosen subtopics, and then prepare and present a report to the whole class.

These are examples of strategies of cooperative learning that have been implemented following the general principles. They can be reused and modified taking in account the particularity of the situation.

3.1.3 Spontaneous cooperation

Taking in account the benefits of cooperative learning, it should be implemented in educational contexts. However, reflection about how to implement it shall also be developed. For instance, Hargreaves (1994), a defender of these strategies, considers that these should be included in the repertoire of teachers, however they should be used with flexibility and discretion, recognizing that their introduction in schools and classrooms constitutes a safe simulation, artificial and bureaucratic, of the forms of collaboration more spontaneous that are possible among students, which have been somehow eradicated by the school and teachers, through discipline control and assessment practices. In this scope, forms of spontaneous cooperation are of great value and unpredictability as the locus of control of cooperation is in the student.

'These forms of cooperation can be named spontaneous cooperation and are characterized by the positive interdependence between participants in order to develop the task, usually present in more structured cooperative strategies referred in the previous paragraph; and by the locus of control of cooperation to be with the students and not with the teachers. In this scope, there are no formal instructions nor rules about how to organize the cooperation as there are in teaching and learning cooperative strategies. These forms of spontaneous cooperation can take a divergent or convergent modality' (Bidarra et al, 2021, p.3) adapting Anderson (2018) distinction that consider divergent play in which children are still mainly centered on their own interests, even if they are playing with other children and a convergent play, which assumes a clear collaboration between peers. In this scope, in a convergent cooperation, participants cooperate to develop an object or product common to the group, whereas in divergent cooperation, spontaneous cooperation occurs but each participant develops its own activity or product.

Final considerations

Summing up, cooperative learning is a classroom methodology and structure which differs from individualistic and competitive structures, served by various strategies that can be developed within the framework of the PUNTE project, and that favor perceptions of self-efficacy, promoting task-centred learning and an attributional pattern oriented towards valuing effort (see Chapter Motivation and engagement on learning).

The implementation of cooperative learning in the mathematics classroom implies that the teacher properly structures the learning environment, carefully choosing the methods to be adopted. Thus, to be effective, cooperative learning activities need to be intentionally addressed. When we prepare a group of students for a cooperative maths learning activity, in essence, we are preparing a team for a game. Poly-Universe is an excellent and effective tool for gamification with cooperative group work in the maths classroom (see also Chapter Guided play).

Without prejudice to the use of cooperative learning strategies that can be used within project activities, it is also of interest to analyze spontaneous and unforeseen forms of cooperation, and how they relate to the dynamics of the activities and other elements of the context, such as habitus, students' age and classroom arrangement.

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3.2 Play, playful learning and guided play

The role of play in human development has been a recurrent topic of discussion and also controversy, among teachers, educators, researchers. This is evident when trying to define what is play, how it can be related with development or learning, which are indicators of playful learning, how it can be implemented with educational purposes. In this text, these concepts are addressed as well as a possible pathway for a playful pedagogy through guided play, namely in maths learning.

3.2.1 What is play? A complex and multifaceted concept

The definition of play has been discussed over the years by different authors, and is therefore a concept that may differ according to opinions and perspectives (e.g. Mardell, Wilson, Ryan, Ertel, Krechevsky & Baker, 2016). In fact, the concept of play is complex and multifaceted, 'the word play alone can conjure up a variety of images, feelings, and activities: to play an instrument, play house, or play with a fish on the line (Mardell et al., 2016). In an overview of the usages of the concept, Smith and Pellegrini (2013) referred that 'Play is often defined as activity done for its own sake, characterized by means rather than ends (the process is more important than any end point or goal), flexibility (objects are put in new combinations or roles are acted out in new ways), and positive affect (children often smile, laugh, and say they enjoy it)'. This is in line with Mardell et al. (2016, 3) when they referred 'Play is typically considered a pleasurable, spontaneous, non-goal directed activity that can include anticipation, flow and surprise Play is both objective and subjective, comprising qualities of observable behavior as well as qualities of felt experience. To characterize play is also important to understand what playfulness is. Playfulness has been considered 'the essence or spirit of play' (p.19), researchers suggest that in order to truly play, children need to demonstrate a predisposition to perceive an activity as play. Christian (2012) summarizes this idea when referring that 'It is the child's playfulness that renders an activity play. As such, playfulness can be seen as the disposition to frame or reframe a situation to include possibilities for enjoyment, exploration and choice'.

Also, play differs from other activities. Following Smith and Pellegrini (2013) play differs from exploration (focused investigation as a child gets more familiar with a new toy or environment, that may then lead into play), work (which has a definite goal), and games (more organized activities in which there is some goal, typically winning the game), the latter usually subject to rules. In some languages, the word with the widest range is 'ludus', which refers to both play and games. Also, although play is a concept that has been linked to childhood activity, it has been progressively explored throughout life. While the nature of play may change as children grow there may be more complex games with rules, advanced physical activity like team sports, programming with computers — the active engagement and meaning-making continues (Frost, Wortham, & Reifel, 2012).

Last but not the least, although maintaining their general features, different types of play have been considered. Smith and Pellegrini (2013) considered five types of play: locomotor, social, object, language, and pretend. Following Hirsh-Pasek, Golinkoff, Berk and Singer (2009) focus on four types (object, pretend, physical/rough-and-tumble, and guided play). In this text, this last type of play, guided play, will be analyzed in more detail, due to the contributions it has brought to the convergence of play and learning.

3.2.2 Why is play important? Play, learning and development

Play has long been recognized as a central way children learn by prominent authors as Dewey (1859–1952), Froebel (1782–1852), Montessori (1870–1952), Piaget (1896–1980), Pestalozzi (1747–1827), Vygotsky (1896–1934). 'Piaget played a central role in the development of the view that play may be of crucial importance in children's cognitive development. Piaget's theories about learning emphasized the need for children to explore and experiment for themselves. For Piaget, play was a means by which children could develop and refine concepts before they had the ability to think in the abstract (…) For Vygotsky, play was also important for an individual's cognitive development, (….) highlighting the social and cultural aspects of play. He argued that during play children were able to think in more complex ways than in their everyday lives, and could make up rules, use symbols and create narratives' (The Open University, 2016).

The idea that play helps children to understand the social world is still current today and there is evidence that it offers a pathway for intellectual, social, emotional, and physical development. Mardell et al. (2016) presented a state of the art about the benefits of play, considering these dimensions. In this scope, research points to the positive effect of play on *intellectual development*, as it fosters engagement and stimulates sense making, allowing learners to build. domain-related skills, content knowledge, and creative thinking. Play also promotes *social development*, 'when learning through play, children often engage with others and make sense of relationships (…). They learn to read cues, listen, and take another's perspective (…) build friendships based on trust and experience the satisfaction of creating with others. (…) to share ideas, express themselves, negotiate, and reach compromises'. When playing, *emotional development* is also promoted, as play may foster self-regulation and children's sense of agency, the capacity to influence, manipulate, and shape one's world. Play may support *physical development* 'as children often choose to play with and through their bodies' and in this scope develop strength, muscle control, coordination, reflexes, and gain a sense of their own body's abilities and limits.

Beyond the overview about benefits of play, Mardell et al. (2016) also proposed indicators of playful learning, with the aim to map 'how playful learning looks like' ( )**.** In this scope, three overlapping categories were identified namely choice, wonder and delight and may represent psychological states as well as observable behaviors. The category *choice* includes 'a sense of empowerment, autonomy, ownership, spontaneity, and intrinsic motivation, while 'wonder' points to the experience of curiosity, novelty, surprise, and challenge, which can engage and fascinate the learner. Finally, delight includes feelings as 'excitement, joy, satisfaction, inspiration, anticipation, pride, belonging.

3.2.3 Guided Play & Maths learning

The concept of play is often limited to younger students who show less academic effort, but play can be a useful strategy in a discipline with greater difficulties experienced by students such as mathematics.

Early childhood teachers understand the importance of play in their students' lives. Free play is spontaneous as children chase their creative fantasies. While, on the one hand, free play is aimed at children and without adult intervention, on the other hand, guided play takes place in an intentional environment that has been carefully planned to stimulate and support children's curiosity and creativity. As students interact with each other and with the materials, teachers observe and record this information to plan next steps, it is the children who decide how they will explore and interact with the materials, not the teachers. In guided play, learning opportunities can be explicitly structured, but the activity is conducted by the child.

The pedagogical objectives of the guidance are therefore broader, in addition to helping children to master knowledge or skills, guided play also aims to provide children with the opportunity to enjoy, control and reflect on their own learning process, that is, guided play is interactive, it is dynamic.

This is an intermediate learning and discovery approach between instruction and free play (Golbeck, 2001). Teachers are seen as collaborative partners who create flexible, interest-driven experiences that stimulate children's natural curiosity, active engagement, and ‘meaning-making’ processes (Fisher et al., 2012). In such contexts, adults protect children's learning by commenting on discoveries, playing with children, and creating games or activities with well-planned curriculum materials.

There is some evidence of this approach with playful experiences with children in maths learning to suggest that children engage in maths-related activities during free play (Bjorklund, 2008), and that certain forms of play are associated with success in mathematics (Ramani & Siegler, 2008; Ramani et al., 2014). Playing with shapes may not lead to discovering their properties or definitions (for example, all squares have four angles). If there is some guidance from an adult, an exploratory conversation can be beneficial to children in promoting learning during guided play (Weisberg & Zosh, 2021). Studies reveal that the level of adult orientation directly influences discovery-learning outcomes (Honomichl & Chen, 2012). Alfieri, Brooks, Aldrich, and Tenenbaum (2011) found that didactical instruction had a greater impact on children's learning outcomes than unassisted discovery.

Studies show the children in guided play improve the acquisition of geometric knowledge (Fisher et al., 2013) or numbers (Weisberg et al., 2013). According to Ramani et al (2014), playing with blocks can contribute to children's mathematical development, such as spatial reasoning, knowledge of geometric shapes, numerical knowledge, and problem-solving skills. As Tian et al (2018) says, playing with blocks is an activity recognized as an effective way to promote children’s overall development, literacy skills, social skills, mathematical skills, and spatial skills.

Guided play can be an important way to support the exploration of mathematical concepts. Engaging with maths-related toys allow children to explore verbally and non-verbally areas such as patterns and shape, classification, numerical knowledge, and spatial reasoning. Studies demonstrate the importance of the nature of maths-related play in preschool-age children. These findings lay an important foundation for understanding the nature of maths-related play in early childhood classrooms and how best to support quality experiences to encourage maths practice and learning.

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3.3 Gardner's multiple intelligence theory

Howard Gardner first presented his theory of multiple intelligence in 1983, in his book 'Frames of Mind'. He defines intelligence as

- capacity to solve various problems in life;
- ability to acquire new knowledge for understanding new information;
- a set of skills which is useful to produce things or to serve the community.

He wrote that there are many different types of intelligence, and learning procedures can be done through these different components of intelligence and on various levels. He believed that people have got nine unique types of intelligence, and everyone possesses all of them in different amounts. So, everyone's cognitive system is built upon a special combination of the nine components.

The nine types of intelligence according to Gardner are the following:

- **Bodily-kinesthetic intelligence:** the ability to manipulate the body or objects with a sense of timing. With this type of intelligence people are able to concentrate on a strong body-mind union. This can be seen in many forms of physical activities, such as dancing, doing sports, as well as in the precise and steady movement of surgeons and mechanics.
- **Existential intelligence:** people with this type of intelligence are sensitive to questions about the meaning of life or death, although they can make rational explanations to philosophical problems.
- **Interpersonal intelligence:** normally every human being is able to communicate with others, but people with higher interpersonal intelligence are able to 'read' others mood, sensitive for nonverbal communications, understand other peoples' perspectives and motivation. They are the 'core' of a community, they have a stronger social sensitivity, they often work as social workers, teachers or actors.
- **Intrapersonal intelligence:** people with this intelligence have the ability to understand their own feelings, thoughts, and motivations. Sometimes they seem to be 'shy', but this means that they are self-motivated, and can use their understanding to lead their life and appreciate other people. Philosophers, psychologists, priests often show high intrapersonal intelligence.
- **Verbal-linguistic intelligence:** the ability to express ourselves in words, with the help of the language (written or oral) is common in mankind. But people with high verbal intelligence use this with more expressions, with a deeper understanding of the meaning of words, in more complex phrases. They are the 'story-tellers' of the community, good speakers, they can use the language to show the different senses. We can see this intelligence in writers, journalists, poets, politicians.
- **Logical-mathematical intelligence:** with this intelligence people have good reasoning skills, abstract thought, recognizing patterns, connecting and categorizing objects according to common properties. So it is not just the ability of calculating, but the logical way of thinking also. They have a good skill for analyzing complex objects or procedures, and can put the details into their right order/places. With experience and special interest they can become scientists, mathematicians, detectives, engineers, computer programmers.
- **Musical intelligence:** they have the ability to reflect on sounds acutely, distinguish between special tones, and are able to repeat rhythms properly. People with musical intelligence are often good listeners, and can reproduce patterns and procedures accurately. This type of intelligence has a similar thinking process with logical-mathe-

matical intelligence. Naturally, singers, composers, conductors, musicians show high levels of musical intelligence, but often scientists and mathematicians also.

- **Naturalistic intelligence:** these people have a sensitivity to all features of nature, they feel responsibility for the Earth's future, and they understand the value of the natural environment. They can become biologists, but also farmers, gardeners, or vets.
- **Spatial-visual intelligence**: people with this intelligence can manipulate images, have good graphic skills, and are able to make good special reasoning. They can understand the relations of 3-dimensional space, and visualize abstract concepts. Naturally, visual artists show a high level of this intelligence, but also architects, sailors, pilots of planes or car drivers need to have it.

Gardner states in his theory that all types of intelligence can be developed through experiences and learning, though it is the person's characteristic which field of intelligence is the best. Also, they can be weakened or ignored if the person doesn't use or practice some of them. To plan an effective and successful learning procedure it has the benefit to recognize and identify the students' weaknesses and strengths. Doing this, the teachers can give appropriate activities to the students to develop the different types of intelligence.

How can we use multiple intelligences in the learning – teaching process? There are many possible methods and formats of organizing a lesson that takes into account the childrens' different types of intelligences:

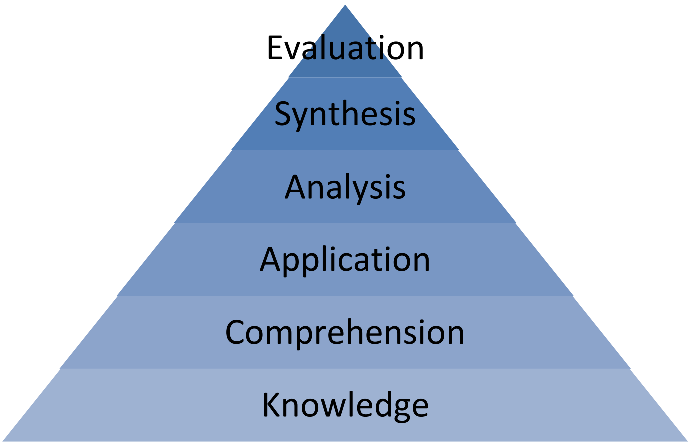
- Use cooperative and collaborative work forms regularly – give different tasks to the group members, according to their intelligence.
- Give open questions, problems that raise their creativity.
- Offer the children the opportunity to work together and to learn from peers.
- Let them work independently!
- Offer the children the opportunity to choose from tasks that need different intelligence.
- Allow them to choose the topic of their interest in project based teaching methods.
- Integrate their interests into the curriculum, connect it to the given teaching material.
- Allow multiple presentations of the products of project work (where different intelligence types can be shown).
- Encourage children to reflect on their own assignments to group work / project work, using multiple intelligence.
- Deliver the learning material in various forms: videos, lectures, discussions, manipulative actions, situation games, computer games, interactive learning materials, etc.

To build up a lesson, where the multiple intelligence theory is taken into account, we first have to identify the children's types of intelligence. To do that, we can use a questionnaire, where there are 10 different statements for each type of intelligence (except existential intelligence), and the students have to give numbers from 1 to 4 (1 – not at all, 4 – definitely true) to each statement. The result of the questionnaire shows us the strengths and weaknesses of the children, and helps us give them the appropriate tasks in a lesson.

3.4 The stages of knowledge in Bloom's Taxonomy

When Benjamin Bloom presented his theory about 'Taxonomy of educational objectives: cognitive domain' in 1956, he did not know that it would be one of the most influential works in American education, and had a great effect on cognitive psychology also. The elegance of the theory and the simple, clear structure makes it usable for any levels of education from elementary school to university courses. According to this theory, the acquisition of knowledge takes place on six orders, beginning with knowledge, through stages of comprehension, application, analysis, synthesis, and going up to evaluation as the highest order of learning. Most commonly it is represented in a pyramid.

The order of the original Bloom taxonomy is the following:



Knowledge: Memorizing verbal information, remember, but not necessarily fully understand the material.

Comprehension: translating into own words, representing, summarizing, able to solve simple tasks with the learned concepts and processes.

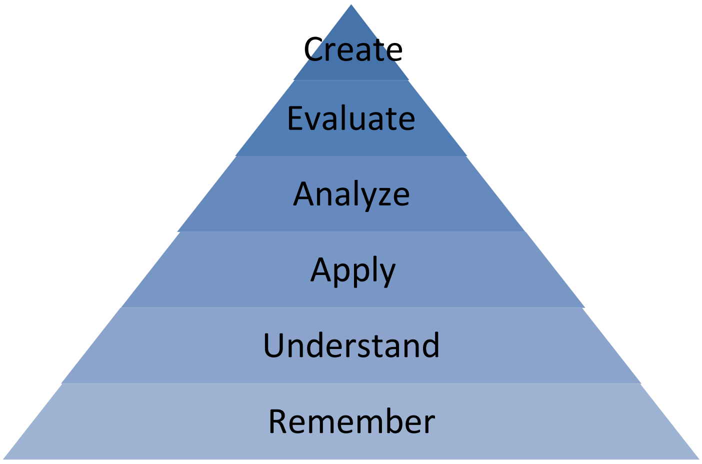
Application: able to solve problems, transfer for abstract or theoretical situations, finding connections and relations with other concepts.

Analysis: Able to identify elements, understand relations between them, determine logical or semantical arrangements.

Synthesis: Combining information to a system or structure, able to create new products, thinking about problems originally.

Evaluation: making decisions, understanding the value of the product, reflecting on your own views.

As we see, the original taxonomy used nouns to name the levels of learning, and it was used as an assessment tool by educators invariably till the 1990's. In 1990's it was revised by experts, led by Lorin Anderson (one of Bloom's students). The revised version uses verbs instead of nouns to name the orders of learning, to emphasize a wider use of the taxonomy in a more activity-centred conception of the teaching/learning procedure. It was published in 2000, and the revised orders goes the following:



A basic understanding of Bloom's taxonomy, and how to apply it, make it easy to clarify learning objectives, adapt technology and develop classroom activities for any grades from elementary school to higher education courses.

Though it was originally invented for assessing higher educational courses, because of its simplicity, easy to understand, its utility and universality, we use it on different levels of education as well. But we shouldn't forget that it is a theoretical abstraction, and not a natural law. It's got its critics also from cognitive psychologists to educators during the years: it doesn't take into account the constructive perspectives of learners, some criticize the hierarchy of the stages, that doesn't allow the learners to analyze and find connections between concepts without comprehension and memorizing basic facts. It is also a matter of critique, that according to the taxonomy, one always has to start from the bottom, to the top, and cannot jump stages – though we know that some students do it. Nowadays, a more flexible representation of the stages can be found, forming a circle or gear *(see e.g.: https://www.bloomstaxonomy.net/).*

Nevertheless, getting acquainted with Bloom's taxonomy gives a useful tool for teachers both for building up the learning material, and for organizing differentiated activities in the classroom. Now we concentrate on differentiated activities. Due to the diversificated knowledge, skills, abilities or motivation of the pupils in one classroom (we call it heterogeneous group) we can split the tasks according to their Bloom-taxonomy level of that special topic.

What kind of tasks can a pupil fulfill on the different stages of the taxonomy?

Remember: able to recall concepts, rules, and recognize them in simple situations.

Understand: able to explain correlations, interpret them in own words, able to represent concepts, rules, understand the connection between them.

Apply: able to recognize and solve simple problems, where the learned concepts, procedures, rules must be used, self confidently.

Analyze: able to solve more complex problems that need splitting into smaller tasks, able to plan the solution and go through it, able to explain the solution to others.

Evaluate: able to synthesize the knowledge, evaluate the results and form your own opinion on it.

Create: able to present new problems, find new ways of representations and new routes of solving the problem. Able to build new knowledge from the present knowledge without any or with little help from the teacher.

As we see, the tasks the pupils, being on different levels, can do is also built upon each other. Using these levels, the pupils can do their own tasks based on their present level, they can reach the next development stage successfully, and it gives a further motivation to learning. We wouldn't say that all the pupils can reach the 'create' stage in all topics. But building their knowledge step by step, always trying to reach the next level, gives the chance to as far as we are able in a real school situation. It takes time, proper planning and consistency from the teacher.

3.5 Combining Bloom's taxonomy with Gardner's Multiple Intelligence Theory: the activity matrix

If we combine the above detailed two theories into an integration matrix, we will get a very useful tool for planning a personalized cooperative classroom activity, where every pupil can find his/her own interest on his/her own level. The integration matrix is a table applicable for combining the interests, intelligence and level of knowledge of the pupils in the classroom. We can place every pupil into it even for a single lesson, according to their current performance. If we see that the pupils go upwards in the table lesson by lesson, it shows the development of their thinking skills and knowledge also.

Intelligence Level	Verbal-linguistic	Logical-mathematical	Musical	Bodily-kinesthetic	Natural	Spatial-visual	Inter-personal	Intra-personal
Create								
Evaluate								
Analyze								
Apply								
Understand								
Remember								

The teachers' task first of all is to identify the current place of a child in the classroom in the matrix. For that, we can use a Gardner test in the beginning of the school year – as we will see, and as Gardner pointed out, every child will show interest, maybe in more types. We can decide in a concrete teaching situation which one of these can be developed through that topic. Maybe, in another lesson, the other type of his/her intelligence factors will be developed.

The next step is to identify the current level of the pupils performance according to his/her previous development on the topic. If we find the place of a pupil, we can give him/her the most appropriate task to be successful, and to develop knowledge. This kind of task dedication is also motivate for the children.

What kind of tasks can we do for the pupils in the different cells of the matrix? We show some examples:

Intelligence Level	Verbal - linguistic	Logical - mathematical	Musical	Bodily - kinesthetic	Natural	Spatial - visual	Inter-personal	Intra-personal
Create	write/tell a story imagine	find another solution, prove	advise, conduct, plan repertoire	plan movement, advise, teach the others,	make action plan, make predict, discover	plan, construct, build, imagine, find out	plan, build, advise, organize, settle	plan, create, stand hypothesis, settle

Evaluate	recast, make conclusion correct, edit	evaluate, recast, measure categorize estimate	evaluate, correct, teach the others	decide, measure, choose, correct	judge, decide, prove, define, balance	evaluate, choose, judge, advise	decide, judge, make conclusion, approve	prove, evaluate, advise,
Analyze	interpret, combine, analyze, try out, take a report organize	analyze, interpret, organize, divide, ask questions, measure, explore	interpret, categorize, organize, make difference	categorize, explore, make order, analyze movement,	explore, combine, categorize, make difference, make experiments	analyze, make an order, categorize, represent, combine,	organize, make a survey, ask questions, categorize	make a test, examine, make controversial, combine,
Apply	make an interview, dramatize, represent, explain	test, solve, count, represent, try out, show the solution	practice, represent, make a show, sing, play, perform	present, perform, act, make it move,	show, represent on diagram, illustrate, use, make a model	dramatize, show, perform, construct, build, illustrate	apply, interpret, make interview, discuss, ask questions	examine, combine, take a look, solve, make a plan
Understand	discuss, reformulate write down, explain, review, transform	write down, define, identify, make sets, count, draw, put in order	recognize, draw it with words, translate, express, sing, play	express, play, put in order, discuss,	place, explain, identify, represent, draw	explain, present, draw, make a framework	write down, explain, discuss, make a report	explain, translate, reformulate, reflect, make an overview
Remember	remember, define, write down, collect	recall, collect examples, name it, list, repeat	remember, recall, repeat, name it, recognize	repeat movements, copy, follow,	define, point, repeat, copy, make memories, choose, fill in	watch, observe, redraw, copy, walk round, reconstruct	repeat, define, name it, collect, tell, show	repeat, name it, examine, remember, recall, evocate

In practice, we can use this method mostly in cooperative or collaborative working forms, or in individual working forms.

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- https://textbookequity.org/Textbooks/Orey_Emergin_Perspectives_Learning.pdf
- https://cft.vanderbilt.edu/guides-sub-pages/blooms-taxonomy/
- https://www.bloomstaxonomy.net/
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3.6 Creativity in mathematics education

The science of psychology was the first to attempt to conceptualize and study creativity at the phenomenon level. The results of psychology lead to pedagogical tasks, which entail a renewal of the methods and tools of education. Since creativity relates to problem-solving and problem-posing, these general educational implications also have specific challenges for mathematics education. In this section, first, we analyze the general notion of creativity and then address creativity from a specific point of view of mathematical education. In this part we describe why the Poly-Universe kit is a creative tool in educational practice.

3.6.1 Creativity in general

The definition of creativity varies from one scholar to another; some define it as a product, others as a process. Stein (1953) defined creative work as new work that the group accepts as sustainable, beneficial, or satisfying. Torrance (1965) described creativity as a series of processes that allow individuals to identify challenging problems, develop solutions and hypotheses, test and retest those hypotheses, and communicate the results. Guilford (1950) approaches creativity from personality and has defined the characteristics and aptitudes of a creative personality. These are fluency, flexibility, originality, elaboration, problem sensitivity, and the ability of redefinition (Landau, 1971).

Fluency means the free accessibility of the available resources (words, thoughts, ideas, concepts, relationships) to the person, making rapid associations according to the circumstances. Therefore, fluency factors are word fluency, thought fluency, associational fluency, and expressional fluency.
Flexibility refers to the ability to transform the available resources, the transferability of stored knowledge.
Originality refers to how rare an individual's idea is in a given situation. This rarity can be measured by the frequency of occurrence in a homogeneous population. Originality often reveals an aspect of a problem that others have overlooked.
The elaboration factor refers to the skills needed to turn an idea into a concrete plan.
Problem-sensitivity means that creative people are particularly alert to unusual things and notice where the problems are. Sensitivity is based on openness to the outside world.
For Guilford, the ability to redefine means using sources in unusual ways.

3.6.2 Mathematical creativity

One of the most challenging aspects of investigating mathematical creativity is the lack of a clear and agreed definition of the term. Most studies of creativity follow one of two paths: extraordinary creativity or everyday creativity (Sriraman, Haavold, & Lee, 2014). Extraordinary creativity is defined as outstanding items that alter our vision of the world. Ordinary or everyday creativity is applicable in a traditional educational system. For example, a new, innovative product or procedure that affects a research area in some substantial manner would be creative for a professional mathematician. At the same time, an unusual solution to a school mathematics problem or an unusual problem formulation could be creative for a mathematics student in school. Most definitions of creativity, whether ordinary or extraordinary, incorporate some degree of usefulness and novelty. However, what is useful and new is determined by the context. For example, the criterion for useful and novel in mathematical science would be very different from what is considered useful and novel in a high school mathematics curriculum. As a result, there is a relatedness aspect to creativity.

In the context of school mathematics, one can say that mathematical creativity is the process that leads to a creative solution or concept to a mathematical issue or the formulation of new problems, which are all valuable to keep. Haylock (1987) emphasizes two elements of mathematical creativity. One is the ability to go beyond the rigid elements of mathematical problem-solving. The other is the ability to think divergently. The majority of creativity studies have utilized divergent output as an indicator of creativity. While divergent thinking is essential for creativity, too much divergent thinking may lead to an excess of novelty at the cost of usability. Convergent thinking must operate in combination with divergent thinking to align ideas with the conventions and knowledge.

3.6.3 Teaching creativity

Although creativity, like most cognitive functions, has a genetic component, it can be fostered and nurtured. What can we do to encourage creativity in classroom mathematics? In general, a sense of doubt is necessary for initiating the creative learning process. Creativity is a habit, and if we wish to encourage it, we must approach it as a desirable activity. Opportunities to practice creativity are required to build creative habits. Students must be looking at traditional problems in new ways. According to (Sternberg, 2017), teaching creative thinking entails encouraging students to create, invent, discover, predict, and imagine. However, this needs teachers to promote and encourage creativity and demonstrate and reward it. These kinds of exercises can help kids develop more innovative approaches to mathematics. In addition, teachers can boost their students' potential for the essential elements of creativity, namely fluency, flexibility, and novelty, by using problem-solving and problem-posing.

3.6.4 Creativity and Poly-Universe

The Poly-Universe kit also allows developing creative habits in the school environment. A single Poly-Universe item is enough to get started. For example, the description of individual pieces can be a good starting point. What can students tell about a particular Poly-Universe object, and how many properties can they describe? Who can come up with more properties or say something that the others cannot?

The elements provide an excellent opportunity to answer simple questions about the perimeter and the area. For example, suppose the teacher asks what the perimeter of the concave shape on the square element (the green polygon in Figure 1) is, assuming that the side of the base square is 1 unit. In that case, we will have students who will determine the lengths of the sides separately using the ratios. However, it is a divergent thinking if the student immediately realizes that the perimeter of the polygon in question is the same as the perimeter of the base square, i.e., four units.

Figure 1:

If using more than one element, one can play different kinds of puzzles. In many cases, it is enough to talk about the free-form puzzle. In addition, the teacher can ask the pupils to make up their own rules for the puzzle and ask questions about them.

Unconventional use of the elements is also possible, notably allowing pupils to go out into space (see Figure 2).

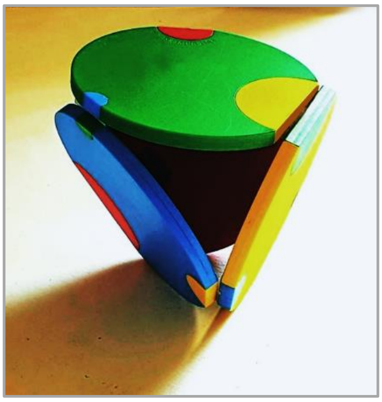

Figure 2: Each item should have three neighbors!

The spatial perspective is something the teacher should also address. Figure 3 shows, for example, a network of an octahedron made up of identical triangular elements. In the glued octahedron, the elements are matched in color and size.

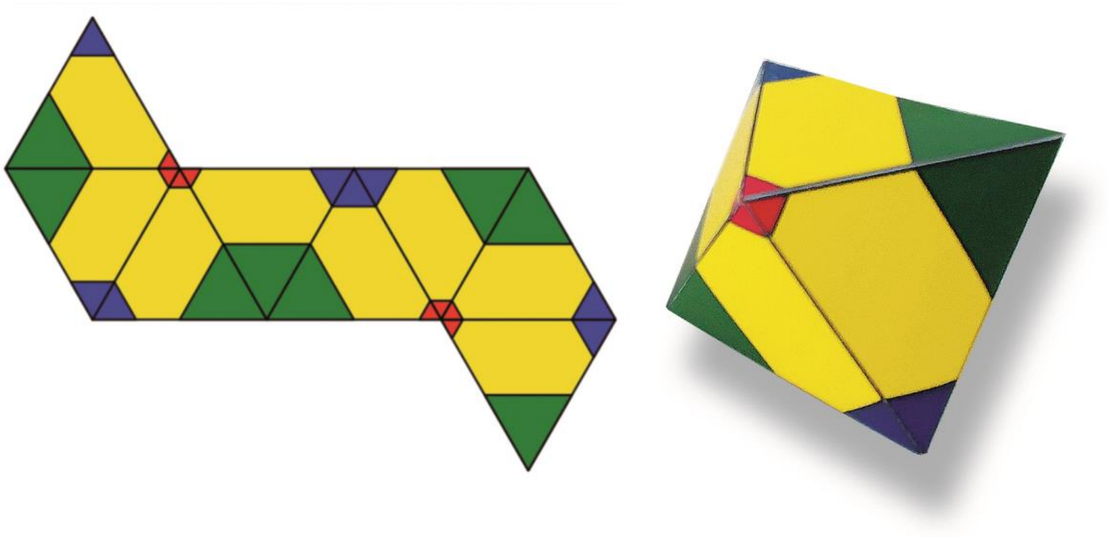

Figure 3. Poly-Universe in space

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3.7 Holistic Approaches and Creative Learning in the Finnish National Core Curriculum

'Holistic' is an often-used term to summarize the essence of Finnish education. Holistic, integrated instruction, along with multisensory, phenomenon-based methods to support transversal competencies, including multiliteracy, are delineated throughout the Finnish National Core Curriculum (FNCC [FNAE, 2016]). Nevertheless, the term *holistic* appears most frequently to describe *holistic well-being. Holistic well-being* and *holistic growth* are complex concepts that characterize the Finnish Curriculum on all levels, including policy, content, pedagogical methods, educational leadership, and implementation. Holistic well-being provides the framework for ensuring safety, physical and mental health, for meeting basic needs, for individual and community care across all levels of Finnish basic education. Holistic well-being and pupil welfare also serve as the background and a goal for building trust, shared responsibility, improving participation, agency, inclusion, the joint reflection of the school and home values, and promoting sustainable lifestyles.

In accordance with FNCC, the core value in Finnish schools is supporting the learning community as the heart of the school culture. The learning community is peaceful and empowering, it relies on self-evaluation and communication with parents and other partners. It is promoting physical and emotional well-being (FNAE, 2016: p. 28.). The establishment of well-being and safety in everyday school life is an important principle in Finnish schools. The school's structures and practices are supposed to create preconditions for learning, equality, flexibility, versatility, accessibility, predictability, fairness, trustworthiness, and warrant for the rejection of discrimination (FNAE, 2016: p. 28.).

The interaction and versatile working approach involve active learning. Learning in Finnish schools is meant to be based on the diversity of learning styles, creative work, play, moving and experiences. FNCC recommends re-connecting formal and informal pedagogical approaches, in-school and out-of-school learning, encouraging project- and module-based education, multisensory learning, and interaction with working life (FNAE, 2016: p. 28–29.).

FNCC promotes cultural diversity and language awareness. According to FNCC, school is at the intersection of local and global perspectives, and part of a culturally transforming and diverse society. This involves the practice of community-based responsibilities and representing multilingualism (FNCC, p. 28.). FNCC supports cooperation between the internal and external actors of education and the society to enhance participation and democratic action (FNCC, p. 29.).

According to FNCC's principles, equity and equality are developed by safeguarding rights, access, and opportunities to fulfill individual needs connected to human diversity and gender equality (FNCC, 30.).

FNCC emphasizes 'ecosocial knowledge' as part of environmental responsibility and sustainable future orientation, through the concept of sustainable everyday life-based wellbeing (FNCC, 30.).

The theme of creativity appears in FNCC nearly 100 times in multiple configurations, contexts, and roles. FNCC, as a policy document, establishes the implementation of creative factors to reinforce the educational system's ecological coherence. The fact that creativity is among the most often mentioned 'cross-cutting' topics in FNCC, serves to validate the concept that the creative nature of schooling is a welcome perspectival change in Finnish everyday education. The processes, partnerships, policies, products and the physical and emotional environment are all included in discussions related to creativity. This completes both the descriptive and transformational capacity of Pamela Burnard's concept of multiple, diverse creativities, see: Figure 1 (Szabó et al., 2021).

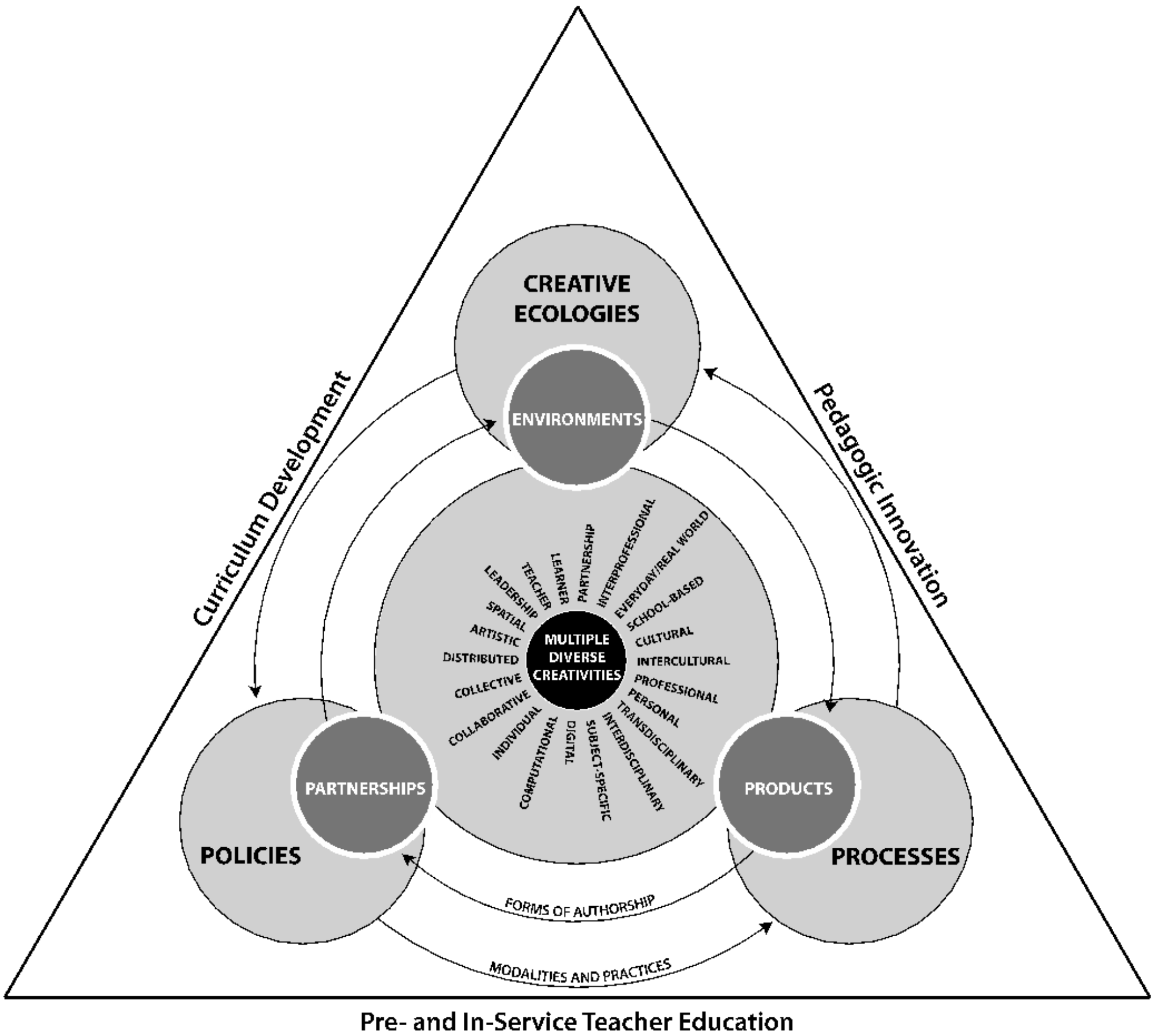

Figure 1. Creative ecologies and micro pluralism of diverse creativities (Szabó et al., *2021*)

Pamela Burnard's model incorporates the dual concepts of 'creative ecologies' and 'multiple creativities.' This model locates the forms, representations, and articulations of collaborative creativity in interprofessional learning and teaching. The model presents *creativities* as multi-stakeholder collaborative *activities,* which are contextualized and reflected by practice-based teacher education, curriculum development, and pedagogic innovations. By highlighting multiple ways of knowing held within a holistic creative ecology, this model's perspectives help shifting the focus from creative capacities or skills to growing creative communities.

Based on Burnard's earlier comprehensive analysis of multiple creativities in music education (Burnard, 2012: p. 223.), the current model also includes reflection on modalities and practices and authorship forms. Just like in the earlier model, the modalities emphasize the role of using several different tools, sharing the production, and blurring the boundaries between formal and informal learning in the collective creative processes. Practice principles can be declared (explicit) or only done without declaration (implicit) and depend on goals and the nature of interaction in communities. In collaborative creativity, the forms of authorship are also subject to dynamic transformation: authorship is negotiated; all actors of the learning process have a role in creating something new; technologies, such as digital tools, can also contribute to the outcome of the creative process (Figure 2).

Figure 2: Collaborative creativities implemented by Finnish students at a Poly-Universe problem-solving session. Photo: Kristóf Fenyvesi

In FNCC, creativity appears in multiple functions. The cultural role and embeddedness of creativity emerge from cultural diversity as a source (FNCC, p. 16.). Creativity also appears in a didactic function, as a source for activities that promote learning, inspire pupils, competence development, and the joy of learning emotional experiences (FNCC, p. 17). Creativity's didactic functions also encourage multiple work approaches, characterized differently in every age group and different learners (FNCC, p. 28.). Didactic creativity is reflected in creative thinking in working methods as well (FNCC, p. 32). Creativity appears in organizational functions and is reflected in the learning environment, which has to offer possibilities for creative solutions (FNCC, p. 30.). Creativity emerges as part of personal, individual characteristics, which education needs to develop in every pupil by developing various skills, including creative communication, like engaging in versatile ways of self-expression and constructive interaction (Figure 3) (FNCC, p. 31.)

Figure 3: János Szász Saxon and Zsuzsa Dárdai leading a Poly-Universe workshop in Finland at the Christian School of Jyväskylä.
Photo: Kristóf Fenyvesi

To grasp ‘everyday creativity’ at work in Finnish schools on the policy level, it might be worth taking a closer look at creativity's role in the didactics of the school subjects, offered for different age groups. One can find a progressive and cumulative plan for creative development embedded in FNCC on the subject learning level as well.

It is important to notice that FNCC's creative development plan organically builds on the work defined by the Finnish National Core Curriculum for Early Childhood Education and Care (FNAE, 2019) and overarching the whole period of basic education in Finland from the first grade to the ninth grade.

According to FNCC's recommendations, the creative development in 1–2 grades is explicitly focusing, but not limited to

- language education and literature to develop verbal expression and imagination (FNCC, p. 110.) and support dialogic, collaborative approaches to cultural expression (FNCC, p. 117.). This involves the creative ways of learning the second national language – Swedish or Finnish – (FNCC, p. 133) and the foreign languages (FNCC, p. 135.);
- mathematics education through creative problem-solving (FNCC, p.139.);
- religion education through pupil-centered, creative methods in connection with ethical questions (FNCC, p. 145.);
- music education through ‘creative production,’ see: ‘The pupils' creative thinking and aesthetic and musical understanding are promoted by providing them with opportunities to compose and perform musical ideas and to use their imagination and creativity both independently and together with others.’ (FNCC, p. 151.). It is interesting to notice that the most enhanced and detailed creative development plan seems to be recommended for music education in this age group;
- visual arts learning through creative applications (FNCC, p. 155.);
- crafts education to develop creativity in close correspondence with spatial, motor and design skills (FNCC, p.156.) and finding creative solutions (FNCC, p. 157.).

According to FNCC's recommendations, the creative development in 3–6 grades is mainly focusing, but not limited to

- transversal competences, especially to ‘Thinking and learning to learn’ (T1), as pupils are encouraged to use their imagination in finding creative solutions in learning (FNCC, p. 165.);
- language education and literature according to similar tendencies as it was seen in the case of 1–2 graders above (see FNCC, p. 173., p. 178., and p. 236.);
- mathematics education through the development of the pupils' logical, precise, and creative mathematical thinking (FNCC, p. 252.) and creative problem-solving as it was seen above in the previous age group (FNCC, p. 255.);
- environment studies through finding opportunities for pupils to experiment, invent and be creative together (FNCC, p. 258.)
- music education, where the theme of ‘creative production’ in this age group grows into a full module of creative development based on complex objectives (FNCC, p. 283. and p. 285.);
- visual arts learning similarly as above;
- craft education, see: ‘Making crafts is an exploratory, inventive, and experimental activity in which different visual, material, and technical solutions as well as production methods are used creatively […] The pupils develop their spatial awareness, sense of touch, and

> manual skills, which promotes motor skills, creativity, and design skills. […] Various transversal themes are studied comprehensively while creating natural connections to other subjects.' (FNCC, p. 290.).

According to FNCC's recommendations, the creative development in 7–9 grades is mainly focusing, but not limited to

- transversal competencies, including 'ICT competence,' regarding information management, inquiry-based and creative work online, and safe digital interaction and networking (FNCC, p. 304.);
- language education and literature in supporting pupils to become 'creators of culture' (FNCC, p. 311.), in addition to the goals, which were already listed in the case of 3–6 graders above (FNCC, p. 322., p. 349.);
- mathematics education as it was seen as above (FNCC, p. 402.);
- chemistry learning through critical and creative thinking (FNCC, p. 424.);
- history learning through familiarizing with the significance of autonomous culture and identity in 'creating, building and defending Finland' (FNCC, p. 447.)
- music education as it was seen as above and enriched with new components, such as developing a creative relationship with music (FNCC, p. 454.);
- visual arts learning through utilizing information and communication technology and online environments creatively, critically, and responsibly (FNCC, p. 458.);
- craft education as it was seen as above (FNCC, p. 462.);
- home economics, to develop manual skills and creativity as well as the ability to make sustainable choices and act sustainably in the daily life at home (FNCC, p. 470.), which also means to 'be creative in the household' (FNCC, p. 471.).

3.7.1 The Poly-Universe Toolkit in the Context of the Finnish National Core Curriculum

The Finnish Experience Workshop STEAM Network's *(www.experienceworkshop.org)* contribution to the *Poly-Universe in School Education* Erasmus+ project offered opportunities to several Finnish students from kindergarten age to higher secondary school to use hands-on manipulatives in the development of logical reasoning, mathematical problem-solving, and computational thinking. Also, to incorporate and reflect on the concept of embodied cognition in practice. This project contributed to shift the focus of holistic pedagogy to key competence development (EC, 2019) through STEAM.

Members of Experience Workshop already have a broad history in implementing the Poly-Universe Toolkit (PT) for symmetry education in mathematics and the arts. The Erasmus+ project has given the opportunity to deepen the knowledge, think further and conduct several experiments in classroom implementation together with Finnish teachers and students, and adapt the PUSE-methodology to the Finnish National Core Curriculum (FNCC, 2014). In addition to focusing on the classroom-based development of various PT-related tasks and projects, we created professional development projects for teachers (Figure 4–5) and introduced the PT at several informal learning events.

Figure 4–5: Experience Workshop's Poly-Universe professional development program for teachers in Finland. Photos: Nóra Somlyódy (top), Kristóf Fenyvesi (bottom).

Based on the interactions with Finnish teachers, during and since the realization of the Poly-Universe in School Education (PUSE) Erasmus+ project, and according to the requirements emphasized in the Finnish National Core Curriculum and included in the PUSE Methodology (PUSE, 2019), we can summarize PT's applicability in Finnish basic education, as follows. PT has a unique potential to introduce and establish the practice of 'hands-on' mathematics or learning-by-doing within a mathematics class (cf. Perusopetuksen, 2014: p. 128, 234, and 374.) because it supports: actions to enjoy 'doing' the mathematics; the ability to consider different points of view in problem-solving and discussing mathematical topics; making mathematics touchable, less abstract, to understand the importance of applied knowledge (cf. Perusopetuksen, 2014: p. 236.); doing mathematics by involving the students' actions based more fully on their cognitive potentials, develop their motoric and spatial skills (cf. Perusopetuksen, 2014: p. 129.), sense of direction and location (the need to use their problem-solving, hands-on and community skills all at the same time); learning mathematics through active playing.

PT has unique potentials to develop skills in the geometric construction (cf. Perusopetuksen, 2014: p. 376.) and scientific categorization of different shapes and objects based on measurements and constructive problem-solving (cf. Perusopetuksen, 2014: p. 236.). PT can be playful and exciting by discovering the geometry of the circle; the square; the triangle. PT demonstrates the potential for developing mathematical/computational thinking by implementing strategic/heuristic approaches in problem-solving. PT offers more hands-on, and trial-and-error-based or open problem-solving paths than it is usually in formal textbooks by implementing conceptual thinking about: mathematical concepts (circle, numbers, square, etc.) combined through activities; making connections between concepts and visualizations; connections between mathematics and the arts by implementing procedural thinking in symbolic procedures.

PT has a unique potential in multidisciplinary/STEAM learning (symmetry, mathematics and arts) + e.g. introducing mathematical concepts in a foreign language. PT has been successfully implemented in learning mathematical concepts in English, when language learning is embedded in non-verbal/hands-on activities; discoveries in Mathematics and Arts connections and various aspects of symmetry in the following topics: Geometry & Measurement (grades 1–12); Combi-natorics & Probability (grades 1–12); Sets & Logic (grades 1–12); Graphs & Algorithms (grades 1–12); Complex Problems & Visuality (grades 1–12).

Artistic connections become important in learning by implementing PT in making own art, symmetry-related compositions out of geometric shapes; thinking about mathematical equations in visual ways; studying art and symmetry to discover mathematical and scientific notions; discovering the role of colors and shapes in symmetry; enhancing the visual impact of mathematics classes; making mathematical concepts come to life through imagination and creative activities.

School implementation of PT for children of ages 6–18 is supported by: easy-to-follow instructions on how to use; easy-to-follow collection of problems for students and teachers; the online instructions and collection of problems is available in English for free; the online materials are easy to print. PT supports various learning styles, both on the individual and group level. PT is applicable even for children of kindergarten age to become familiar with basic shapes and geometrical aesthetics and basic notions of symmetry through simple games.

PT can support learning-together activities also in an extra-curricular and an out-of-school learning context. The material of PT is very durable. PT is not only an educational tool but also a fun game, it can support a positive attitude towards learning and can increase motivation and engagement (cf. Lukion, 2014: p. 130.).

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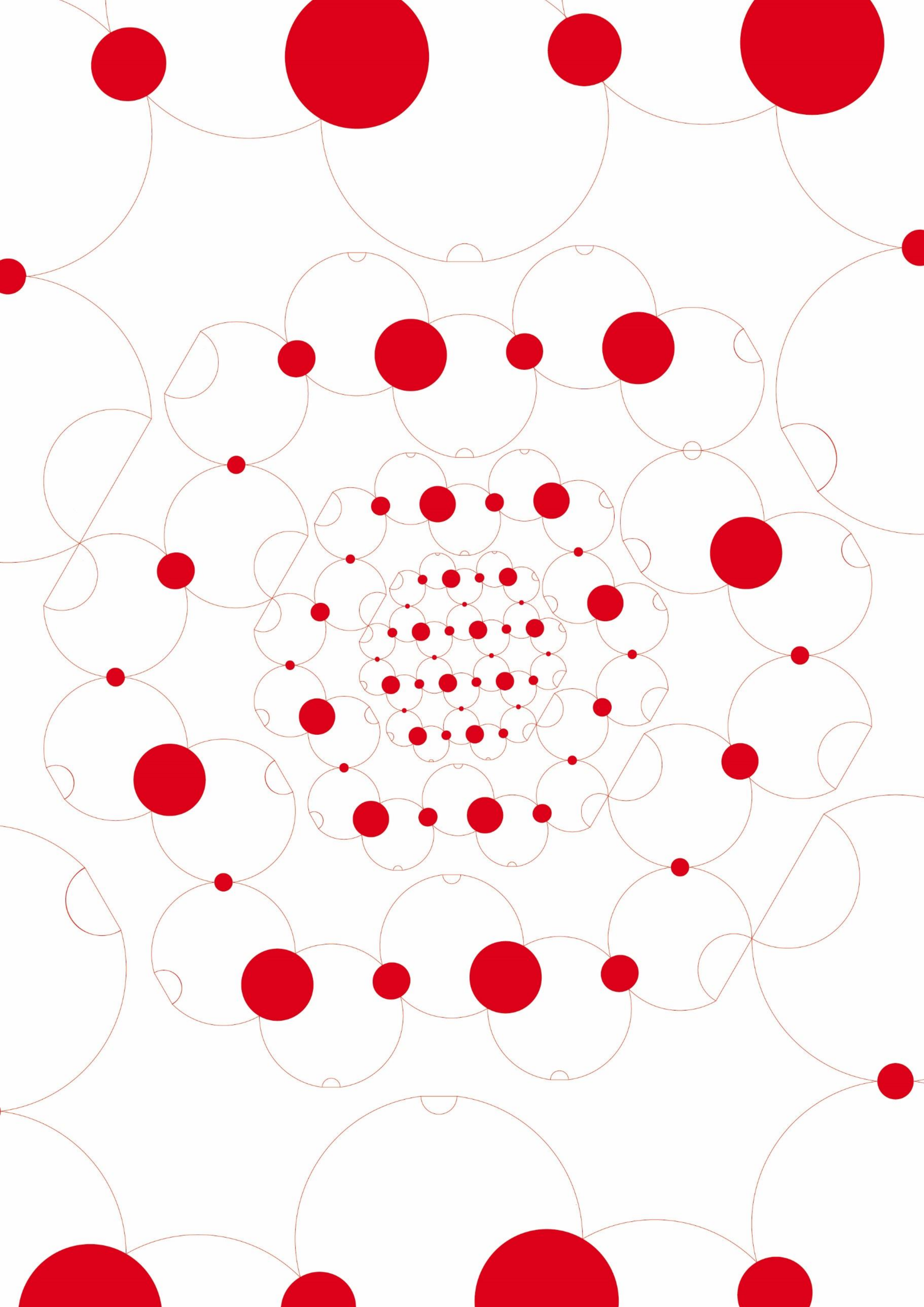

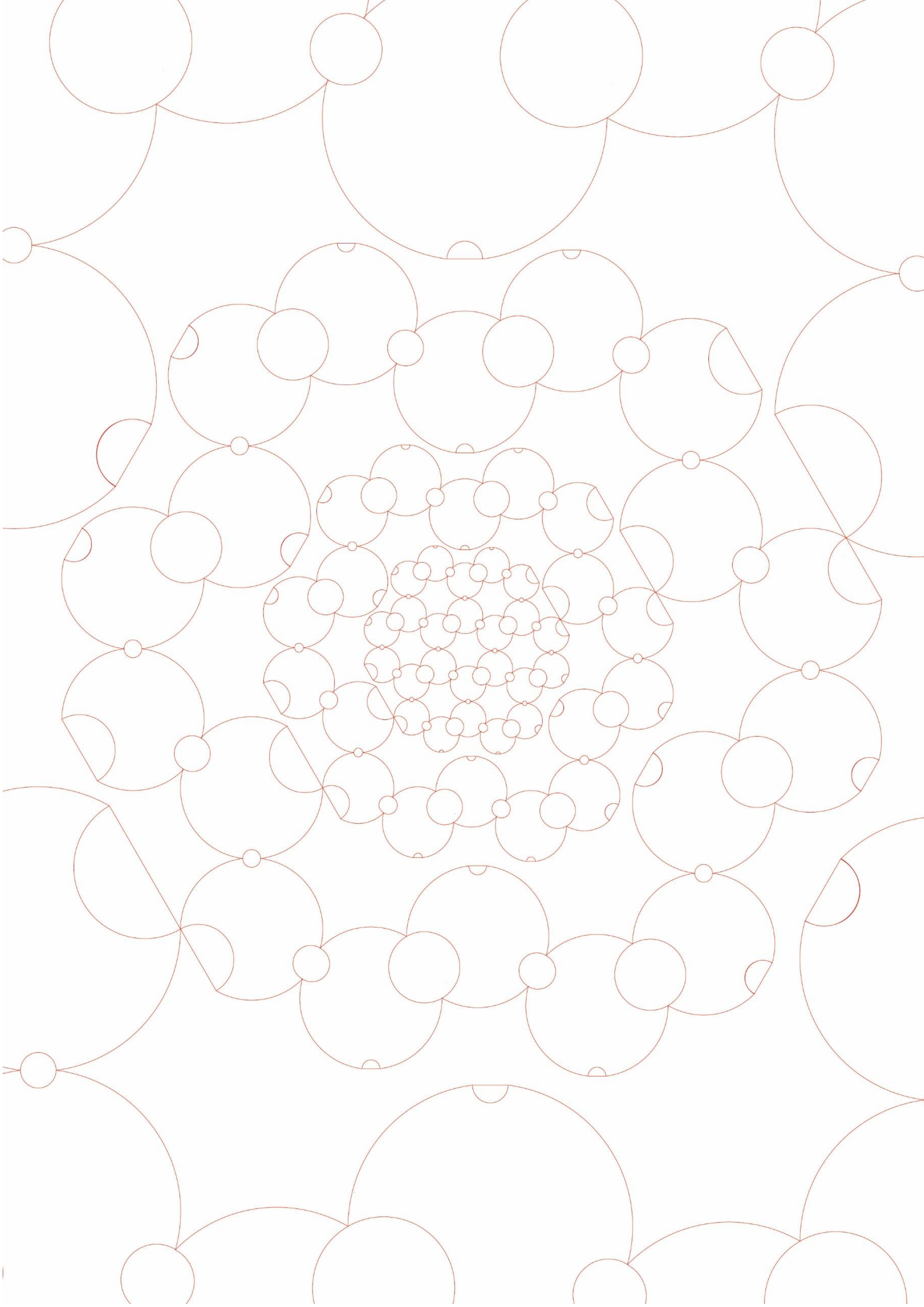

IV PLACE AND TOOLS OF POLY-UNIVERSE IN TEACHER TRAINING

There are many different ways and approaches of organizing the teaching-learning process in real practical situations. In the previous chapters we have described – without the need of totality – various theories and methods behind a well-thought-out, valid, up-to-date way of teaching. The teachers' most powerful weapon is the wide range of methodological background, from which we can choose the most appropriate way of presenting the learning material to the children. There are also many different tools we can use to bring the teaching material closer to the children, or to develop thinking, learning, social, personal, communication, etc. skills and abilities, as well as the key and transversal competences. One of these tools can be the Poly-Universe game. We have shown the background of the game family, and some ways it can be used in different situations of teaching-learning process. We have also shown some skills and abilities that can be developed through the Poly-Universe game. Based on these, we think that this game family has its own place in teacher training education. In the following chapter we would like to describe the realization of it – showing the possibilities of using the game in formal and informal learning contexts, the fields of the training where it can be fitted, and the framework of possible methodological courses, regarding the differences in the level of teacher training education and the national differences in the partner institutes' countries.

4. 1 Poly-Universe in formal and informal learning contexts

4.1.1 Informal learning contexts

The Poly-Universe game family can be used as a 'family game', because of its simplicity, its universality and its interest from the age of 3 to 99. When first meeting the game, we try to discover its properties, the characteristics of the elements, the different ways we can join them, without any given rules. The best thing in this game is, that when you catch it, it raises up your interest immediately, and awakens your creative mind. What can I do with it? What if I… ? And it happens to everyone irrespective of their age! As we know, playing is one of the most important ways of learning for children. They learn many basic knowledge from informal playing situations: communication and social skills, spatial seeing, whole-part perception, algorithmic thinking, senso-motoric abilities, synthetic and analytical thinking, seriality, etc. As we have seen in the previous chapter, these skills and abilities (and even more!) can be developed through playing with the Poly-Universe game family. So it has its place among the Lego, board games, wooden cubes and prisms, parlour games and other logical games in a family repertoire of playing toolkits. It can be taken out time by time, and every single occasion will be a new game to play with.

The simple yet interesting structure and the fact that there are no given rules for playing makes it a universal game – similar to the games of Zoltán Dienes. It depends on the age and the number of players, how we start playing with it. It can be a construction game, but also a parlour game. We can give many different interpretations to the elements. The rules can be simple or complex – it also depends on the age and the general developmental level of the players. As there are no previously given rules, the players can change the rules to make the game more interesting, more enjoyable. Next time we can state new rules, and the playing situation will be more and more complex, as the children grow, and as they have more experiences with the game.

We dare to restrict anyone's creativity, so we wouldn't give examples for family situations (and also, this is not the purpose of the present study), but if anyone needs initiative ideas, can find some in the PUSE Methodology book (http://www.punte.eu/puse-methodology/).

The game family can also be introduced and used on out-of-classroom school events. The Poly-Universe was first introduced in Hungary for a wider audience by the Experience Workshop movement (https://www.elmenymuhely.hu/). Through these workshops many schools (pupils and teachers) got acquainted with a new view of teaching – STEAM – which now has a broad international interest from far Eastern to Western and Northern European countries. Among other activities, the Poly-Universe game was a standard participant of the workshops country-wide. There were several occasions when the Poly-Universe game was introduced internationally at educational and art conferences from Bridges to ICME. You can find the details about these events on our webpage. So for the informal contexts a school, or a teacher has got the possibilities to get to know about the Poly-Universe game, and even to buy the toolkit set (http://poly-universe.com/poly-universe-game-family/).

4.1.2 Formal learning contexts

The inventors, some mathematics teachers and the members of the Experience Workshop team believed that there are more in this game, and it also has its place in formal – curricular learning situations. That's why they've started to develop the PUSE Methodology book for mathematics teaching – as a previous project in an international cooperation of Hungarian, Finnish, Spanish, Slovakian teachers and researchers. They have identified fields of mathematics, where the game can be used: combinatorics, probability calculations, planar and spatial geometry, measurements, graphs, algorithms, set theory and logic. For using the game in mathematics lessons we can find several tasks for different school levels (from elementary to high school education and talent care). These exercises can be used in mathematics lessons organized according to the Dienes-Varga method, or using cooperative techniques. Even at the beginning it was clear that the game can be used for other subjects – mainly for art teaching too. That's why there are tasks named 'complex, visual and interdisciplinary' in the PUSE Methodology book. We can go further on this route, and in the next chapter we will show some examples of using the game in different school subjects and interdisciplinary approaches.

The exercise banks are very good to have – they give a munition for the teacher, where he/she can browse for the appropriate task for a special lesson. The other thing is how we can plan a lesson like this, let's call it a 'complex lesson'. Let's see the details!

A 'complex lesson' is a normal subject lesson in compulsory education, which is built upon some principles, uses methods of differentiation, consists of trans-curricular elements and supports the development of transversal skills. Meanwhile, in the centre there is the learning material, given in the curriculum. The principles are the main principles of 21st century education: learner supportive, adaptive, teaching equity, community building. What makes such a lesson unique? First, the novel approach differs from the ordinary ways, and helps the acquisition of the teaching material. This novelty can come from the teaching methods, the working forms or the used tools. Second, the motivation effect of the lesson, the aim of raising the interest, making the learning material live and exciting. These types of lessons give the pupils a positive charge with which the other, traditional, ordinary lessons (and probably the whole school life) will become more acceptable. A complex lesson develops the pupils' critical thinking, creativity, initiative, problem-solving, risk-managing, decision-making and emotion-handling skills. These are the so-called 'transversal skills', the development of which are indispensable for a complete improvement of the pupils' personality. In these lessons the pupils' activity has an equal role with the content of the teaching material. The complex lessons give a new viewpoint, which combines the learning material with real life, with other subjects and the development of the different components (cognitive, sensitive, social, creative) of personality. The novel, extraordinary approaches, the experience of surprise and the chance of unfolding pupils' different talents make an inspiring classroom environment. In the spirit of complexity we shouldn't stay at the stage of 'one subject – one approach' (e.g. mathematics – logical games, literature – arts, history – stories …), but be brave to mix the viewpoints. Combine history or geography with arts and games like Poly-Universe!

The principle of this kind of complexity in the lessons can also be found in the National basic curriculum of Hungary (2020), and also in the Finnish core curriculum of 2014.

4.1.3 The planning procedure of a complex lesson

We should start the planning of such a lesson at the recommendations of the basic curriculum and the developmental goals formulated in it. From that the main contents of the subject materials are derived. During the yearly planning of the syllabus of our own subject, we can think through the matching points between the content and the game. If we know which skills and abilities can be developed through using a complex approach with Poly-Universe, we will find those lessons that can be organized like this. If we have an idea, don't hesitate to work it out! Creativity and methodological diversity is one of the most important strengths of the teacher. But we don't always have to create a new exercise, if we want to realize a complex lesson. If we just have the feeling that a special content can be taught with the aid of Poly-Universe games we can look for an appropriate exercise in the PUSE Methodology book, or in our 'task-bank'. If we find a good example, we can use it as we like – for a starting point, an advanced organizer, practising, summarizing knowledge, group activity, education or ventilation of a lesson. Don't hang on to realize the chosen exercise or method exactly as it is written in the book, but be free to modify it, adapt to your own pupils' expectations, the classroom environment, and other possibilities. Also, we always have to know the pupils well, to give them a challenging but reachable problem – as we have seen in the problem based teaching method. All this expects creativity and extra work from the teacher, but the result – the good mood of the lesson, the glimmer in pupils' eyes, the better understanding of the learning material, and the development of their skills – compensate for it.

We can see the main steps of the procedure of planning on the following figure:

What shall we be careful about?

- Never let the connection between the game and the learning material be constrained! Choose those contents, where the connection naturally comes from the properties of the game.
- Keep in mind that the main goal is to learn the given material. The forms and tools of the teaching/learning procedure are always subordinated to the syllabus.

- We use a special approach and play the game in order to make a better understanding of the learning material, and not for the game itself. It is now about the formal teaching context, and the daily routine at the school.
- Choose the place of using a complex approach (or Poly-Universe) well in the syllabus. It is better to have every tenth lesson like this, equally distributed during the school year, than once a year for ten lessons.
- Let the pupils get acquainted with the game or method before using it on a 'serious' lesson – as part-time activity or out-of-classroom occasions. It takes some time to come to terms with it, and our aim is to help the acquisition of the knowledge, not to make it more difficult or slower.
- The main focus is on the pupils' activity, their reactions and their advancement.
- Use differentiated activities if possible, and give special tasks to pupils with different talents and knowledge levels according to Gardner and Bloom theory.

If we plan a complex lesson with the Poly-Universe game like this, it will fit well into the school's syllabus, a real advancement can be reached, and the pupils and teachers will both enjoy these lessons.

4.2 The fields of using Poly-Universe in teacher training

As the previous chapters show, there are plenty of opportunities and cases for using Poly-Universe in formal and informal learning-teaching contexts. To be able to do so, students of different levels of teacher training education should learn these methods during their studies. Though it is a special tool, it has got so many connections to modern methods of teaching practice and the development of children's mental skills. That is the reason why we are working out courses for university students from kindergarten education through primary school teacher education to secondary school subject teacher education to get acquainted with the possibilities of using Poly-Universe in their teaching practice.

Our aim is to implement the previously detailed contexts into a modern view, and leading to an experience centered way of teacher training. In the present study the authors collected the theoretical, methodological and artistic background upon which framework of a complex methodological course can be built up. It is also possible to use only parts of the study, and build in its content to an existing university course of general or special pedagogical – psychological studies.

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4.3 Framework of PUNTE methodological courses

The aims of the PUNTE courses are:

- to construct and organize learning environment such a way that enable students to recognize Poly-Universe as a useful teaching tool and to develop their creativity which will help them to create new applications of Poly-Universe in teaching and learning;
- to prepare the students to be able to apply Poly-Universe in teaching using different approaches;
- to help students' learning and overcoming their learning difficulties connected to notions of the modules of the course;
- to develop students' abstract thinking, logical thinking, analytical thinking, spatial seeing, and student's skills of problem-solving;
- to develop students' sensitivity to be more tolerant and emphatic and to use inclusion;
- to develop students' sense of art;
- to help students to handle the electronic version of Poly-Universe.

4.3.1 The structure of the course:

This course is meant to be a one-semester course (duration: 13–15 weeks) with at least 2 lessons per week. If the curriculum is more flexible, it can be 2+2 lessons per week (lecture and practice). In the planned course there are compulsory and optional modules, so that every participating institute can adapt it to its own characteristics of teacher training fields (from elementary teacher training to secondary school subject teacher and engineering, from general psychological approaches to subject didactics). During the semester the students get acquainted with the theoretical background written in the present study and the practical use of the Poly-Universe game in different pedagogical situations, according to their main university programme and levels of teacher training. During the semester the students have to fulfill a special project individually or in teamwork. The topics can either be devised by the students (discussed with the educator) or suggested by the educator. At the end of the course the students present their project results (construction, piece of art, description of special usage, trying out, etc.)

4.3.2 The course consists of the following modules:

1. Introductory classes- exploration of the shapes of the elements of Poly-Universe sets and discussions about their potential usefulness in teaching and learning based on appropriate theoretical background (duration: 1 week)
2. Geometry and the methodology of its teaching (with appropriate theoretical background) where the Poly-Universe is used (duration: 3 weeks).
3. Combinatorics and the methodology of its teaching (with appropriate theoretical background) where the Poly-Universe is used (duration: 2 weeks).
4. Informatics and the methodology of its teaching (with appropriate theoretical background) where the Poly-Universe is used (duration: 3 weeks).
5. Developing abstract thinking, logical thinking and analytical thinking by using Poly-Universe with appropriate theoretical background (duration: 2 weeks).
6. Complex, interdisciplinary problems where Poly-Universe sets are used (duration: 2 weeks).
7. Using Poly-Universe as concrete representations in solving problems (duration: 1 week).

8 Electronic version of Poly-Universe and how to use it in teaching with appropriate theoretical background, GeoGebra applications (duration: 2 weeks).
9 Games in teaching and learning when we use Poly-Universe sets with appropriate theoretical background (duration: 2 weeks)
10 Inclusive teaching aided by Poly-Universe – how can the game help disabled students' learning and their communication to one another and how can it help students and their teachers to be more tolerant and to use inclusion (duration: 2 weeks).
11 Poly-Universe and MADI art movement (duration: 3 weeks)
12 Interdisciplinary approaches of teaching aided by Poly-Universe (duration: 2 weeks)
13 Presentations of students' work and ideas about using Poly-Universe sets in teaching and learning and discussions about them (duration: 1 week).

4.3.3 Contains of modules:

1 During the introductory classes, through exploration of the elements of sets and through playing and dealing with them (spontaneously or guided by teachers) students can recognize some possibilities of using Poly-Universe sets in teaching and learning and can discuss them. Through these activities students will get some ideas of Poly-Universe applications or new variations for existing applications and can easily understand theoretical background.

2 One week students deal with geometric figures which can be constructed using Poly-Universe sets and with their properties. With help of their teachers the students formulate and solve simple and more demanding problems connected to these figures. The students and the teacher discuss the methodology of teaching these contents.

One week students deal with geometric transformations in the plane using Poly-Universe sets and with their properties. With help of their teachers the students formulate and solve simple and complex problems connected to these transformations. They also discuss the methodology of teaching these contents.

One week students calculate perimeter and area of geometric figures which can be constructed using Poly-Universe sets and the area and volume of geometric solids which nets can be constructed using Poly-Universe. They also formulate similar problems and discuss the methodology of teaching them.

3 Basic concepts of combinatorics like permutations without and with repetitions, variations without and with repetitions and combinations without repetitions are explained by using Poly-Universe sets. In the first week, permutations and variations without repetitions are introduced. In the second week, variations with repetitions and combinations without repetitions are explained. The Poly-Universe set is used to improve recognition of permutations, variations, and combinations.

4 The aim of the Informatics module is to use Poly-Universe in teaching programming in Python and coordinate geometry. Basic concepts of Python programming language and some libraries in Python are introduced. The Turtle Python's library is used to draw given patterns. While writing the program for drawing a given pattern built by Poly-Universe set, students will intensively use their coordinate geometry knowledge, and broaden their knowledge in Python programming.

5 The Poly-Universe set is used to practice tasks which will improve logical thinking of students. The students will be encouraged to introduce their own exercises for developing and improving logical thinking.

6 Students deal with complex, interdisciplinary problems where Poly-Universe sets are used. The students learn to analyze the problem and the connections among different parts of that problem. They try to create that type of problem and discuss the possible methodologies of teaching them.

7 Students learn about using concrete and visual representations in teaching and then solve different types of problems where the elements of Using Poly-Universe sets are used as concrete representations. The students formulate such types of problems and discuss the possible methodologies of teaching them.

8 Through using an electronic version of Poly-Universe, the students explore its characteristics. They analyze and compare them with the characteristics of classical Poly-Universe sets. They try to formulate the advantages and disadvantages of using the electronic version of Poly-Universe and to collect problems during which solving this version of Poly-Universe is useful.

9 Students learn about using games in teaching and learning. They play different games using Poly-Universe sets for learning notions of different fields and for students' development. Then the students analyze the games and try to create new ones or to modify the rules of the games they played. Through discussion students try to improve the characteristics of the games and try to find more applications for using Poly-Universe sets through games.

10 Students learn about inclusion and about its importance in education. In concrete examples they learn how to use Poly-Universe sets to create an inclusive learning environment which can help disabled students' learning, communication and collaboration. In order to be more inclusive and tolerant the students play games using Poly-Universe sets, where their eyes are closed to feel what the world looks like in the sense of the visually impaired students. Then the students discuss their experiences and feelings.

11 The students can see some photos of pieces of art which are inspired by Poly-Universe or where the elements of Poly-Universe sets are used as pieces of which the piece of art is constructed. Inspired by these, the students try to make something new and discuss their work.

12 The students get acquainted with the STEAM model of teaching and other interdisciplinary approaches of teaching: how can we ruin the fences between different subjects by playing with the Poly-Universe game. We try out different viewpoints where the game creates a bridge between real life and school subjects, or between different sciences. We create practices to show how we can use the Poly-Universe game to help understand the concepts of different subjects.

Table of compulsory and optional modules is given in Table 1:

Modules	Module’s type
1. Introduction	compulsory
2. Geometry and the methodology of its teaching	compulsory
3. Combinatorics and the methodology of its teaching	compulsory
4. Informatics and the methodology of its teaching	compulsory
5. Developing logical thinking	optional
6. Complex, interdisciplinary problems	optional
7. Poly-Universe as concrete representations in solving problems	compulsory
8. Electronic version of Poly-Universe	optional
9. Games in teaching with Poly-Universe	optional
10. Inclusive teaching aided by Poly-Universe	optional
11. Poly-Universe and MADI art movement	optional
12. Interdisciplinary approaches	optional
13. Presentations of students’ work	compulsory

Table 1: Compulsory and optional modules

EVALUATION

The final evaluation of the course is based on both theoretical knowledge and the presentation of the student's project. The examination consists of simple questions to test basic understanding and knowledge. In the presentation students present their ideas and possible application(s) of Poly-Universe.

4.4 Dynamic GeoGebra Applications inspired by Poly-Universe

Although the proportions of all three elements of the Poly-Universe tool are fixed, János Szász Saxon, the artist who invented the device, also experimented with changing the proportions. Examples of these are shown in Figures 1 and 2.

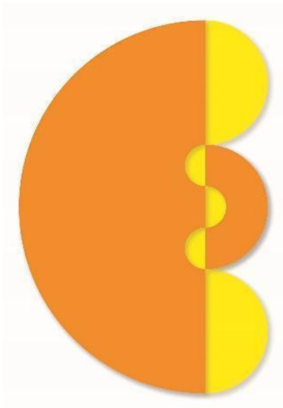


Figure 1: SAXON – King and Queen

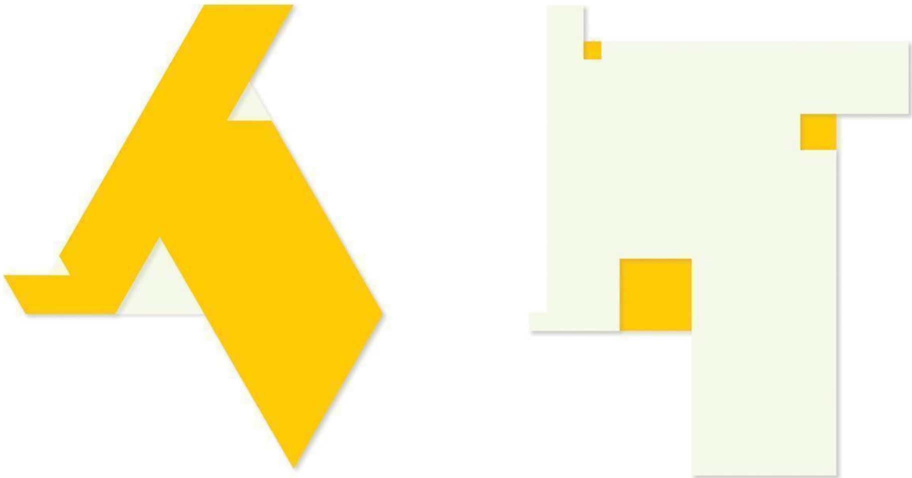


Figure 2: SAXON – Rota-Rota triangle, Rota-Rota square

The GeoGebra software can be used to change the proportions and to visualize them dynamically. We first deal with the editing of the basic elements, taking advantage of the mathematical relationships between the individual details of the elements, geometric transformations, and then we think about changing shapes and proportions in several directions. We change the number of sides of polygons, the ratio of similarity, and we move to spatial shapes. Finally, we illustrate fractals related to the basic shapes of the Poly-Universe and their spatial extension, noting here the affinity with the inventor's works of art. The related edits are collected in a GeoGebra book.
https://www.geogebra.org/m/ms8nzfym

4.4.1 Representation of the basic elements

The basic Poly-Universe shapes of triangle, almost square (hereafter just square), almost circle (hereafter just circle) are easy to represent in GeoGebra. There are several ways to construct triangles and squares. One method uses only regular polygons and bisectors. Much more exciting is the other option, which starts from the basic shape and obtains the vertex shapes with enlargement from point, as shown in Figure 3. The large triangle (AEH) is obtained from the basic shape (ABC) with centre A of ½ ratio, the medium (FBG) with centre B of ¼ ratio, and finally the small triangle (IJC) with centre C of 1/8 ratio by a similarity of centres. Thus, in just a few steps, the entire basic element can be constructed.

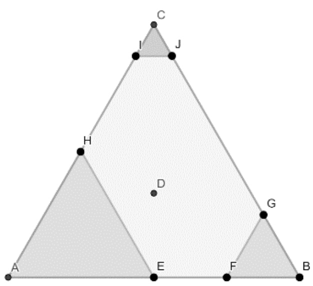


Figure 3

You can achieve the desired colored shapes by the settings of the triangles and squares (color → opacity → 100, style → line thickness → 0). In the case of the square, the hole is also edited and then colored white, Figure 4.

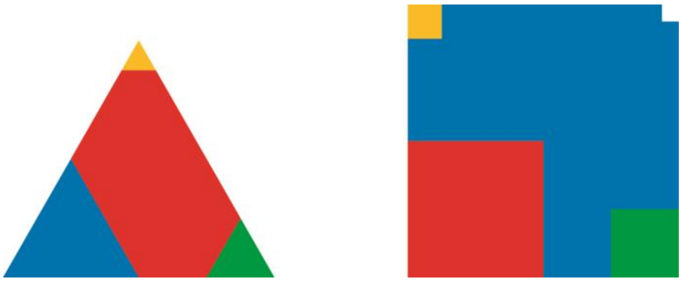


Figure 4

One can also ask the question, how can each large, medium and small triangle or square be derived from each other by what transformations? For example, consider the notation in Figure 3 to derive the medium from the large triangle. The large triangle AEH is rotated by 120° about D, and then reduced by ½ from point B to get the medium triangle BGF. Using GeoGebra's dynamic capabilities, we can do this spectacularly using two sliders. We adjust the sliders so that while on one the angle increases from 0° to 120°, on the other the ratio decreases from 1 to 0.5, Figure 5.

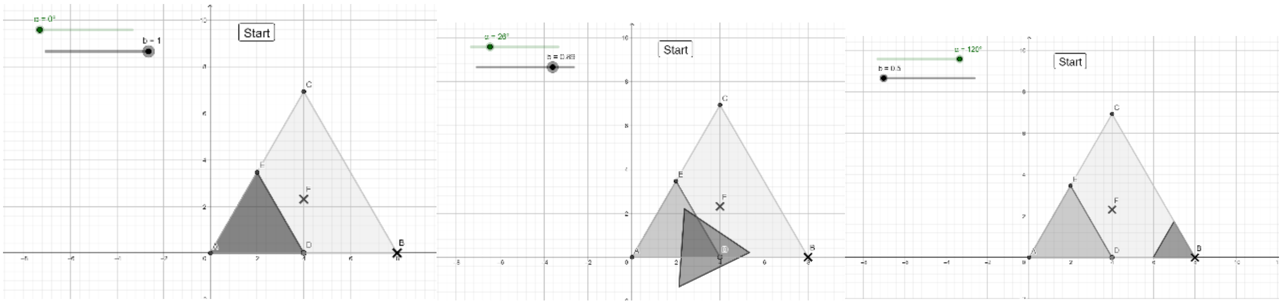


Figure 5

There is no elegant way to edit the circular shape like the previous ones, only we can draw this by applying many small, simple editing steps one after the other. This is illustrated in Figure 6, which shows all the auxiliary lines used for the editing on the left side, and on the right side only the image of the cleaned shape.

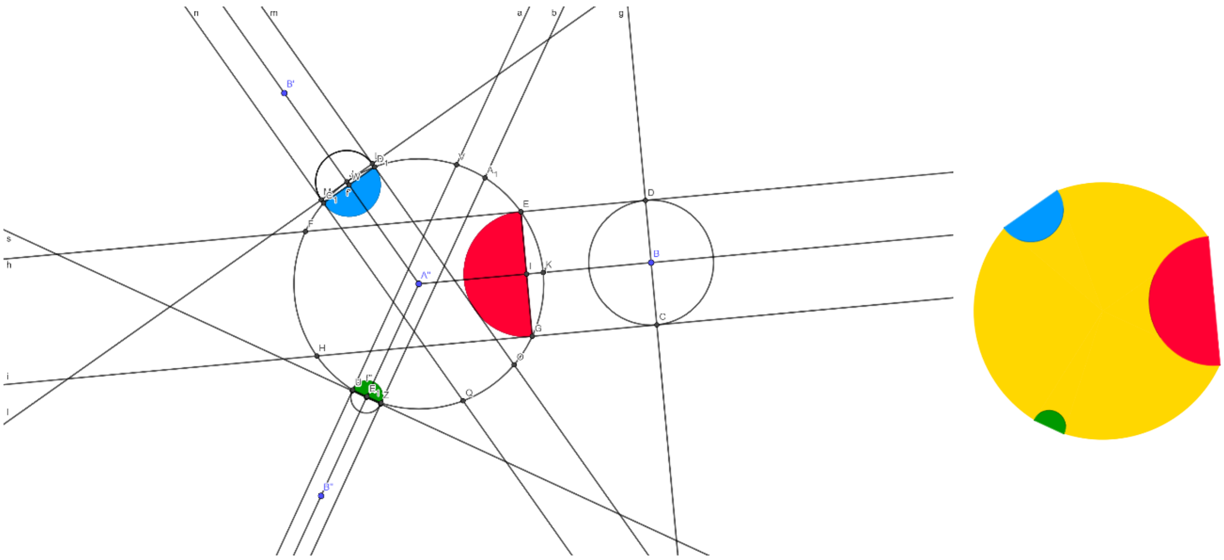


Figure 6

4.4.2 Further thinking and transformation of the basic Poly-Universe forms

Changing side numbers in 2D

One of the questions that might first arise is: what if we start increasing the number of sides, for pentagon, hexagon,...n – angles, we start drawing vertex shapes with ratio ½, ¼, ... $\frac{1}{2^k}$? At first glance, we get a surprising result. In the case of a hexagon, 1–1 sides of the two largest vertex shapes fit on top of each other. And for a larger number of sides, these two polygons also overlap, Figure 7.

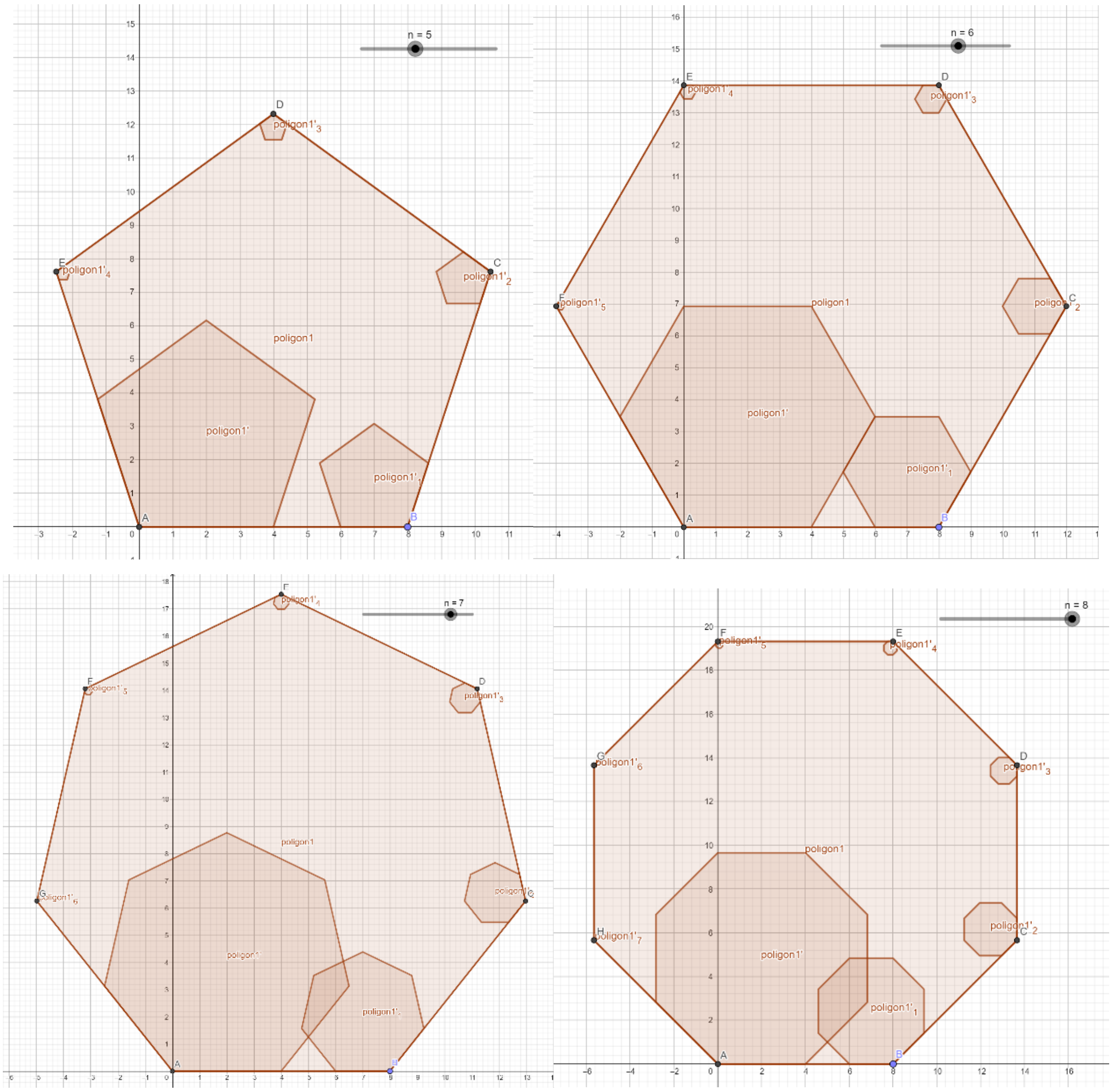


Figure 7

Why, in the case of a hexagon, the 1–1 sides of the two largest vertex shapes fit together can be shown by elementary geometric proof.

But other questions may also arise. Could we calculate the area of the common part of the two polygons? If we were to change the ratio, at what ratio would the two largest vertex shapes have no common part? There is a GeoGebra application for this – you can change the ratio in addition to the number of sides – you can experiment with this and then try to prove it. What if we put the vertex shapes in a different order, i.e. after the ½ ratio we would not put ¼, but say 1/8. Would the above problem still exist?

4.4.3 Changing ratios

In the introduction, we mentioned that János Szász Saxon also experimented with proportions, and not only inwards, but also outwards from the basic form, i.e. with a negative ratio. Figure 8 shows the variations of the ratio ‘b’ of the triangular form in 4 different cases: $-2 \le b < 0$, $0 < b < \varphi$, $\varphi < b < 1$, $1 < b \le 2$, where $\phi = \frac{\sqrt{5}-1}{2}$, the rate of the golden section. The ratio of the golden ratio first came up when we thought of a 3-dimensional extension of the Poly-Universe square. A smaller cube was placed in each of the 8 vertices of a cube, each cube being reduced to ½ of the previous one. From vertex 6 onwards, the cubes were almost invisible. The question arose how far the ratio could be increased so that the two largest vertex shapes did not overlap? We then solve $a = \lambda \cdot a + \lambda^2 \cdot a$ equation for λ, where λ is the similarity ratio and ‘a’ sign the edge of the base cube. The positive solution to the second-degree equation will be the ratio of the gold section. Figure 8 shows the result for 4 substantially different ratios, the bottom right image shows an edit with a ratio $\varphi < b < 1$. The most interesting thing here is the effect of the translucent colors on each other.

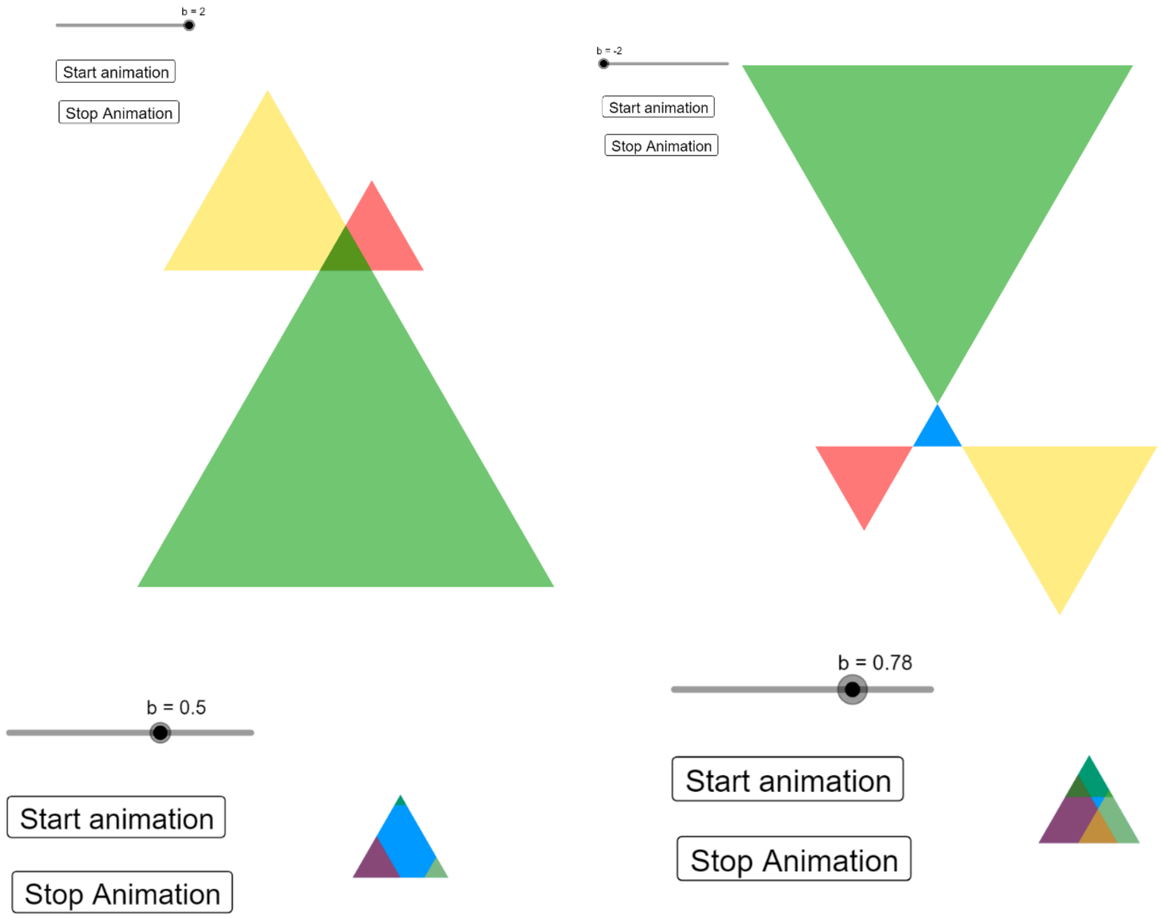


Figure 8

The edits above can provide countless outsights and points of connection. First of all, there are mathematical connections. We can talk about the properties of the enlargement from point for different proportions, the area of similar polygons, but the golden ratio is also related to art as well as mathematics. The ratio of the golden section appears in countless paintings, buildings, music, but also in our everyday objects (e.g. a bank card is a golden rectangle) and in familiar logos.

4.4.4 Poly-Universe in 3D

The 3D extension of the triangle element is naturally the tetrahedron, and the square element is the cube. The enlargement from point in GeoGebra also applies to spatial shapes, so editing is easy and can be done in a few steps. For the tetrahedron in Figure 9, the ratio remained the traditional ½, but for the cube we used the ratio of the golden ratio for the reasons mentioned above. Of course, the base element can be colored, but in the figure the base element has been made with an edge structure for better visibility.

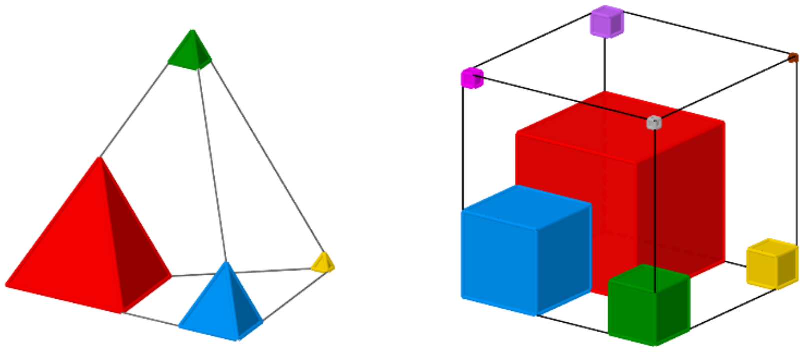


Figure 9

You can also experiment with variable ratios on the slider in the two polyhedron-based cases. Figure 10 shows the tetrahedron-based construction with a negative ratio less than –1 and a positive ratio greater than 1. We can also formulate relations for the volume of similar solids, read the volume of the polyhedra in the algebra window, and study the properties of the enlargement from point in space.

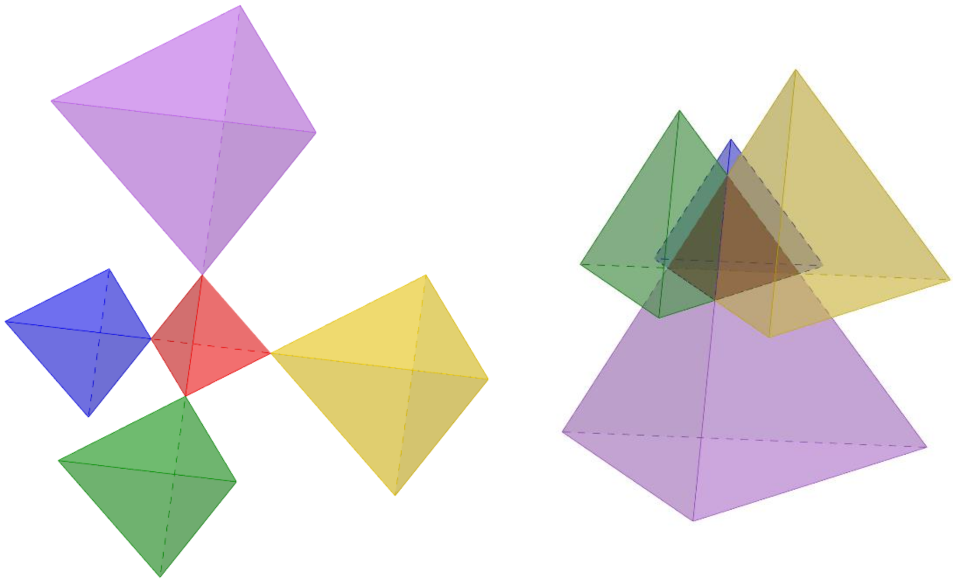


Figure 10

Figure 11 shows a negative proportional similarity of the cube and an interesting projection of the cube with a positive proportional similarity.

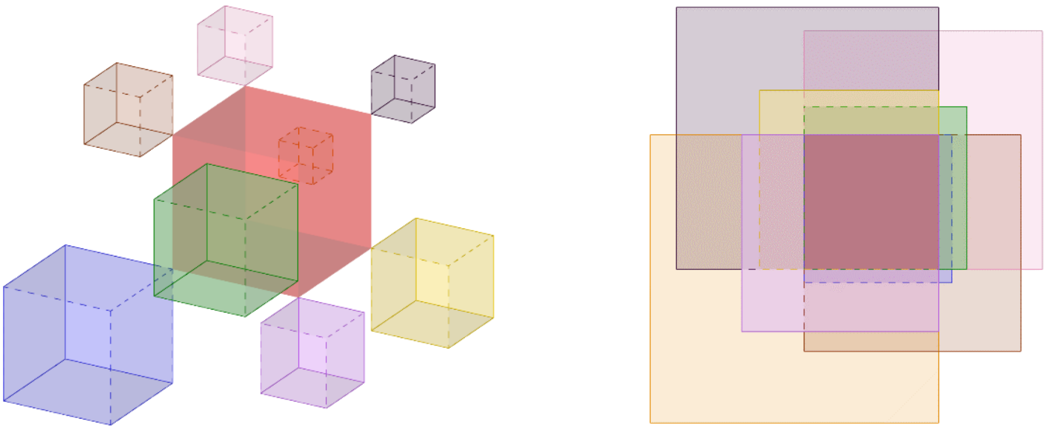


Figure 11

4.4.5 Poly-Universe and Fractals

Beautiful fractals can be created on the computer. But is a fractal made with different software an art? Here is a possible answer from art historian Géza Perneczky:

'Most of the paintings were almost monochrome, because they were made up of combinations of just two colors, cadmium yellow with its strong tone and luminous white fields. But that was not the strange thing, it was that although at first sight the paintings appeared to be very clearly structured Constructivist compositions, following the well-known style of geometric abstraction – somehow they were not! Behind the seemingly simple proportions lay an unusual complexity, and this tension, which was difficult to unravel, was almost tingling. Then it suddenly became clear to me what I was seeing. They were indeed fractals, and this was a sensation, because I had been interested in this relatively new branch of mathematics for about ten years, but as an art historian I could also see that I had never before found a successful example of the creative application of fractal geometry in the visual arts anywhere in the world.'

Géza Perneczky, art historian, wrote the above lines about the works of János Szász Saxon in the publication 'Artist between the realms of mathematics and 'beautiful proportions

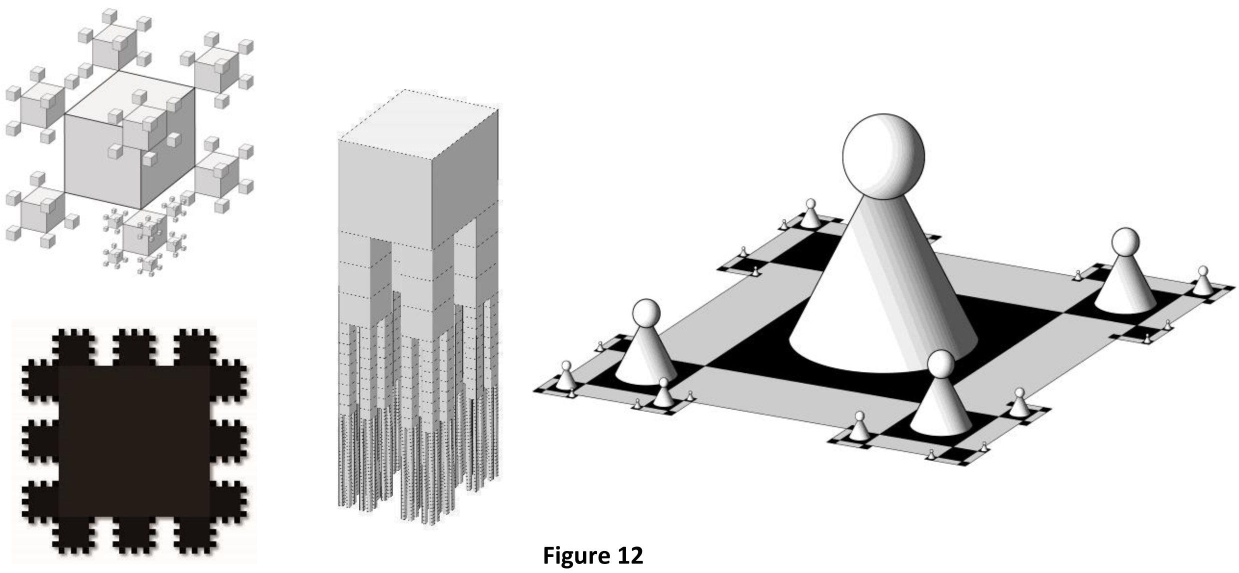
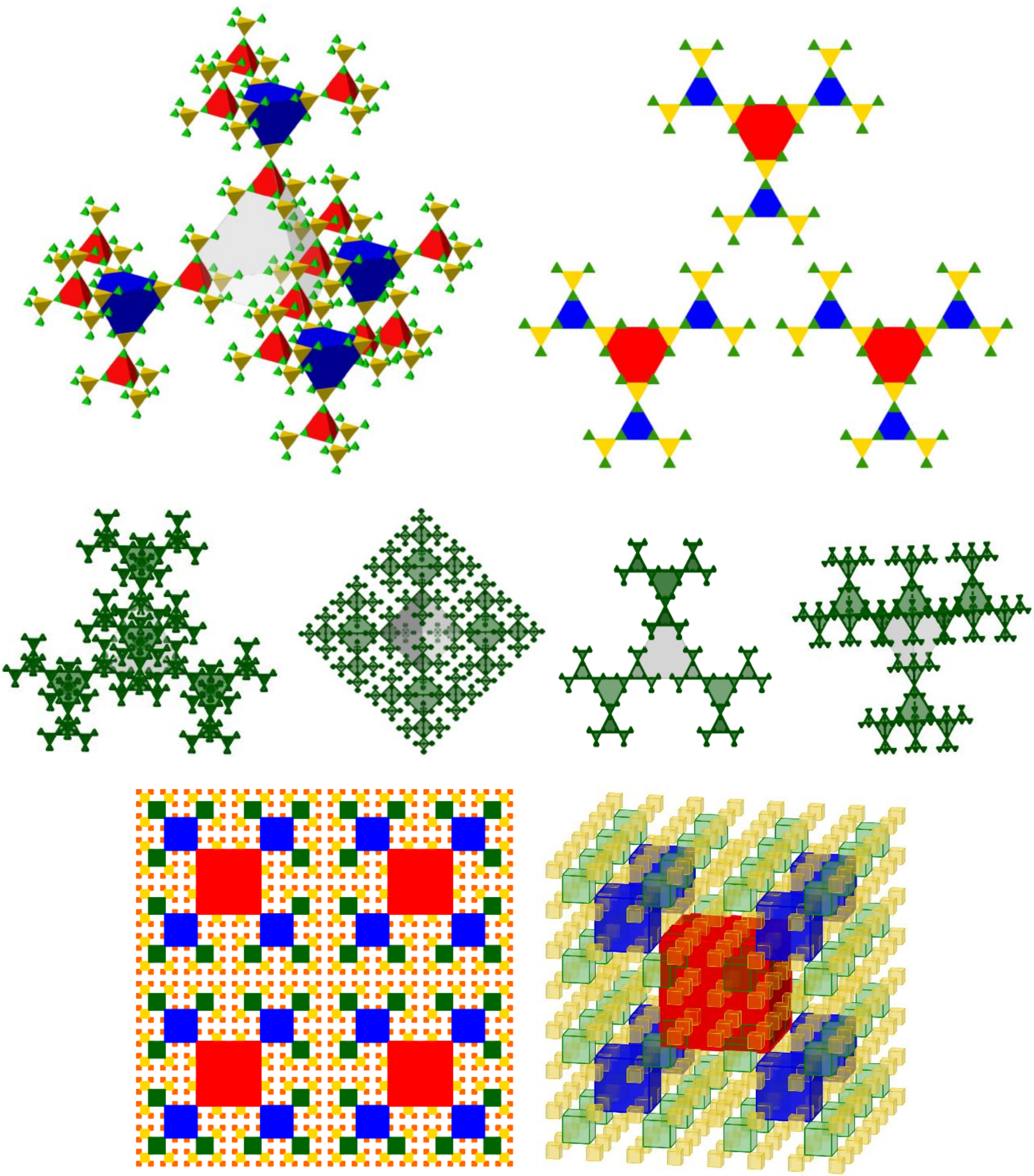


Figure 12

In Figure 12 we see planar and spatial fractal-based works by János Szász Saxon.

In GeoGebra, it is perhaps the drawing and coloring of fractals where we can best express our creative tendencies. These fractals can be colored – corresponding to the Poly-Universe or different from it, monochrome, with different shades of parts, made in plane from triangles, squares, or based on a tetrahedron or cube in space. The projections of spatial fractals can also be very interesting, Figure 13.

In the process of drawing and studying fractals, we can calculate the circumference and area of the overall shape at each step. We can study what the limit of the perimeter or area will be if the number of steps goes to infinity. Can an infinitely long broken line be placed in a finite area? We can explore the concept of dimensions, why are fractals fractional dimensions?



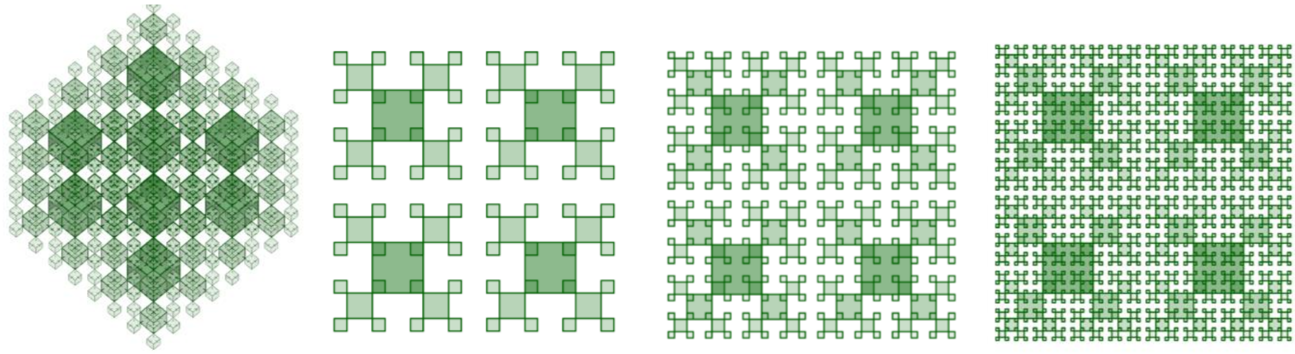


Figure 13: Various triangle, square, tetrahedron, cube-based fractals and their projections

The creation of fractals in the GeoGebra book is very similar. First, you draw the initial shape, which can be a planar polygon (triangle or square in the book) or a spatial polyhedron (tetrahedron or cube in the book). Then we arrange the vertices of the initial shape into a list. In the next step, a sequence of the initial shape is created with the elements of the vertex list as the centres of similarity, of course, the ratio of similarity must be given. This gives us another list. Then we repeat this step as often as we see fit, only always replacing the old list with the new one. The steps copied from the GeoGebra algebra window for a triangle are shown in Figure 14.

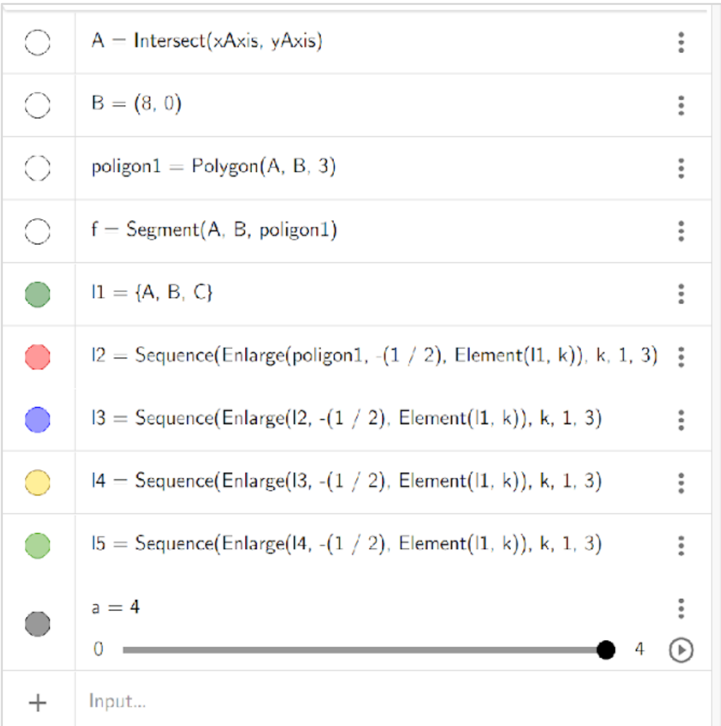


Figure 14

CONCLUSION

Of course, building on the elements of the Poly-Universe, we can also come up with many more ideas, either modelled in GeoGebra or other computer programs, or using the physical tool, or just drawing and thinking on paper. Of course, the mathematician's ideas are primarily mathematical, but artistic connections are also natural, and spatial extensions of the Poly-Universe to the crystal lattice can also provide a way forward. An interesting question is what is the spatial equivalent of a circle shape? Can we construct it in GeoGebra? In addition, the list of questions is endless each new question gives rise to more and more ideas.

4.5 Poly-Universe digital game interface

4.5.1 The PUSE e-learning platform and curriculum roll-out

The PUSE Methodology was developed by educational institutions and teachers from four EU Member States during the Erasmus+ project 2017–2019. The PUSE tasks support the teaching of mathematics in primary school for lower (6–10 years old), upper (10–14 years old) and secondary (14–18 years old) students.

One of the issues in the application of PUSE e-learning is the different educational systems in different countries. It is necessary to be prepared for the fact that third countries have different education systems from the EU, so that EU curricula will not only have to be translated into the language of the target country, but also adapted to the classes of the school system there. This difference between countries also applies to pre-school education, e.g. the US does not have the public kindergarten system as in the EU, but has 'childcare' first and then pre-school a year before children start primary school.

The other most important issue is the research of the Poly-Universe invention comprehensive algorithm with the involvement of universities and Academic of Sciences, should be our first and most important area of development. In the development of the PUSE Methodology, there remain many open, unresolved questions, which can only be answered in a reassuring, scientifically rigorous way by the development of a comprehensive algorithm for each basic form of the tool. The results of the scientific background research and the elaboration of the overall algorithm will be incorporated into any further development, which could provide the basis for e-learning developments and game applications that can be transferred to the practical field of basic education. In addition, these results provide a comprehensive picture of the interdisciplinary issues of STEAM-based educational development, which could also be an excellent methodological basis for higher education.

4.5.2 PUSE e-learning platform and curriculum development

The main unit is the PUSE e-learning platform and curriculum, which will be developed in the framework of the product development of Poly-Universe Ltd. Its aim is to make the tasks of the PUSE Methodology, developed in the framework of the Erasmus+ PUSE project, available to teachers and students in a digital, interactive format. The e-learning module will initially contain 150 mathematical problems from the PUSE Methodology book, which will be accessible and solvable by students through a digital version of the analogue Poly-Universe game on a dedicated interactive e-learning interface.

Teachers will also be able to create and invent new PUSE exercises on their own in a PUSE Editor interface, which will be uploaded to the PUSE Knowledge Base after a quality check and accessible to other teachers, thus ensuring the continuous updating and expansion of the PUSE e-learning curriculum. The PUSE e-learning platform will provide the possibility of collaboration, whereby a group of teachers and students or a group of students can create together on PUSE. The completed solutions and creations can then be shared online on various social networking platforms.

The PUSE e-learning courseware will be created according to the SCORM standard, which ensures that all standard LMS (Learning Management Systems) are able to be imported, depending on the existing system used by the educational institution for distance learning. In principle, e-learning courseware is designed for three types of use:

- support for classroom-based task solving,
- homework assignments,
- incorporation into distance learning materials.

For the above reasons, we are developing the learning materials to be accessible through standard ‘smartboard’ software used in the EU, in web browsers, for both iOS and android operating systems.

One of the main goals of the system is to replace the analogue game elements with an online, universally accessible interface for learning the Poly-Universe game.

The interface provides the user with individual elements (triangles, circles or squares) which can be moved using the ‘drag and drop’ principle. During the movement, the elements that may be connected at each point are ‘contracted’, i.e. the element that has just been grasped is moved to the new position.

The fitting of the elements together can be done in the way defined in the methodology (see PUSE (Poly-Universe in School Education) Methodology for Teaching Mathematics through Visual Experience 2019 *(ISBN 978–615–81267–3–1) – Glossary).*

The playing field is basically a large table surface on which the user can move freely and drag new elements (a given number) onto the surface from two sides (possibly from below).

When the user feels that they are done with the task and can answer the questions asked, they can move further within the interface to enter their answers or attach a finished picture of the solution.

The game interface will also store a so-called ‘history stack’ alongside each solution, which will allow more functionality to be built into the system.

Playing the items stored in this stack one after the other (while keeping a certain interval) will result in an animation that will guide you through each step of the user's solutions.

In addition, it is possible to add ‘Back’ and ‘Forward’ buttons, allowing the user to undo certain steps and then ‘replay’ them if he/she needs to.

The user's actions on a given task are stored, so that when he closes the application and starts solving again, he can pick up where he left off. If the user wants to start the task from the beginning, they can do so by pressing a ‘Reset’ button.

4.5.3 Fragmented e-learning systems

An important element of the PUSE educational package is the digital e-learning courseware, which plays an important role either in classroom work or in the preparation of homework to be done at home. For e-learning courseware, it is important that it is compatible with the LMS (Learning Management System) used by the educational institution. LMS systems are quite fragmented in the world, which means that there are a few LMS systems that have a prominent market share in the world, such as Google Class and Moodle, but in addition there are thousands of LMS systems in use worldwide.

The PUSE e-learning courseware is designed to comply with industry standards (SCORM, xAPI, Cmi5), making it importable into any standards-compliant LMS system.

The other distribution option in the e-learning world is to connect to e-learning marketplaces (e.g. Udemy, OpenSesame, Teachable, Thinkific, etc.). In this context, the question is which of these platforms provide learning material for PUSE's target audience (students and teachers aged 6–18).

Presentation of platforms similar to PUSE

https://www.mathway.com/about
The student provides the task, the program solves it, graphing the functions he/she has given. Limits: it calculates only area, volume, and surface area. It indicates if e.g. insufficient data is given. With over 5 billion problems solved, step by step work, no ads, web and app, Mathway is an 'assistant'.

https://akriel.io/hu-HU
Hungarian development, currently free of charge, but usually there is a fee. This is for grades 5–12, with lots of algebra, but no functions or geometry. You can see the step-by-step solution of each problem, the details of the steps and one can also ask for a detailed explanation. It is a good practice of algebraic problems and concepts, precise, and well structured.

https://www.nkp.hu/tankonyv/matematika_5/lecke_04_001
Covering the whole NAT (including geometry), one can find interactive exercises with checks and videos. Professionally, it is probably the closest to what PUSE is trying to create, but with limited functionality.

https://www.matific.com/hu/hu/home/
This website has a strong pedagogical background (materials compiled by professors, educational experts), reporting on students' work to teachers individually, in groups, or in classes. Progress tracking, personalized pacing, intelligent algorithm, automatic assignment are available. Experience of discovery, problem solving, critical thinking are in the focus. Access is free.

https://www.matekmindenkinek.hu/?fbclid=IwAR2OmFDXT
i14l5VazEqdaMYh__CJZ2z3NIwK9RdQ0Q5_Z434YVrAHav-9M
There is a free-to-use section, but subscription and extra materials are also available. One can find videos about the theory, then very simple problems are solved. The paid and free-to-use parts together cover all the materials for primary and secondary school.

https://www.matika.in/hu/?fbclid=IwAR122m4uRjsQNA2iOjMXgdOokrqBfFyM0KyLGlPqhngMC7w-RKqkH_LazNI
Only for primary school students. Good exercises, interesting, thought provoking, lots of games, against the machine (Matyi) and also games for pairs. It's free, and available in Italian, Hungarian and German. It is run by a company registered in the Czech Republic, based on the method of Milan Hejne, containing thinking exercises rather than template exercises.

https://webuni.hu/intezmeny/matek-online
hu?fbclid=IwAR08AvBlo84z6n5zuQMVG9NHDVKct4nXpJuCxdLVZEjZaSrqAdwr9dsSc9Y
Payment is required. It is for grades 5–12. There are many chapters, geometry included.

https://www.khanacademy.org/math/geometry?fbclid=IwAR2OmFDT-
i14l5VazEqdaMYh__CJZ2z3NIwK9RdQ0Q5_Z434YVrAHav-9M
One can find several subject topics, course materials, practice exercises, detailed explanations of the correct solution in case of mistakes. There is also a translation into English. There is teacher support, which can follow students' work, link to google Classroom. This is a 'development-focused' approach.

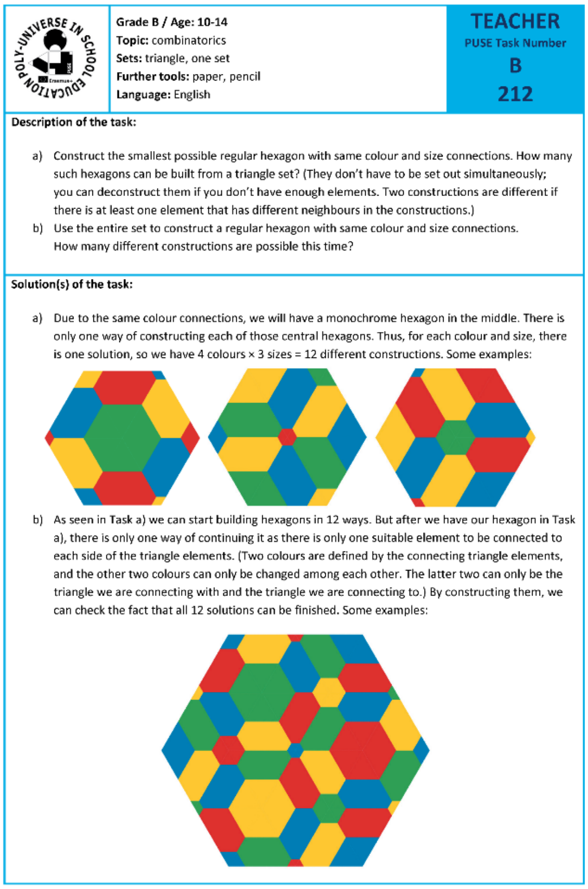
https://www.mozaweb.hu/hu/lexikon.php?cmd=getlist&let=MICROCURRICULUM&sid=MAT
Mozaik is proud to announce its very own Chromium-based web browser called mozaWeb Browser. It was designed and optimized to provide the best user experience while using our interactive content on mozaWeb. It provides the fastest and smoothest experience for us cloud-based educational solutions. 3D scenes open lightning-fast, as it already has they mozaik3D Player plug-in built-in. Generally, it provides a hassle-free user-experience both for students and teachers.

An English site:
https://www.teachthought.com/pedagogy/apps-websites-teaching-math-
online/?fbclid=IwAR0ufXKr2yM9CVGu2T8ig7dhkJM31MP2J26D5G_FUeM8TvEAlBt3gTuzqmk

To what extent is PUSE better than the previous e-learning materials?

Most national e-learning mathematics curricula are almost exclusively about algebra, with geometry being a minor part of the curriculum, only in chapters. So what makes the PUSE curriculum unique? PUSE is all about visually, experiential learning, creativity, interdisciplinary approach, creativity, the possibility to invent new and own problems through the related applications, the possibility to try out the analogue game without being asked. While 1 to 1 tasks are mostly related not only to one subject, but to two or three at the same time, the methodology based on the same family of tools can be applied across the widest spectrum, from kindergarten to university. PUSE is not closed, it is interactive, most of the time the tasks are invented by the children themselves and solved together, while the curriculum is constantly evolving with/by the adults of the future.

The PUSE e-learning platform can be an excellent tool for higher education, teacher training, in addition to basic education, and is therefore essential for the implementation of the PUNTE project.



Grade B / Age: 10-14
Topic: combinatorics
Sets: triangle, one set
Further tools: paper, pencil
Language: English

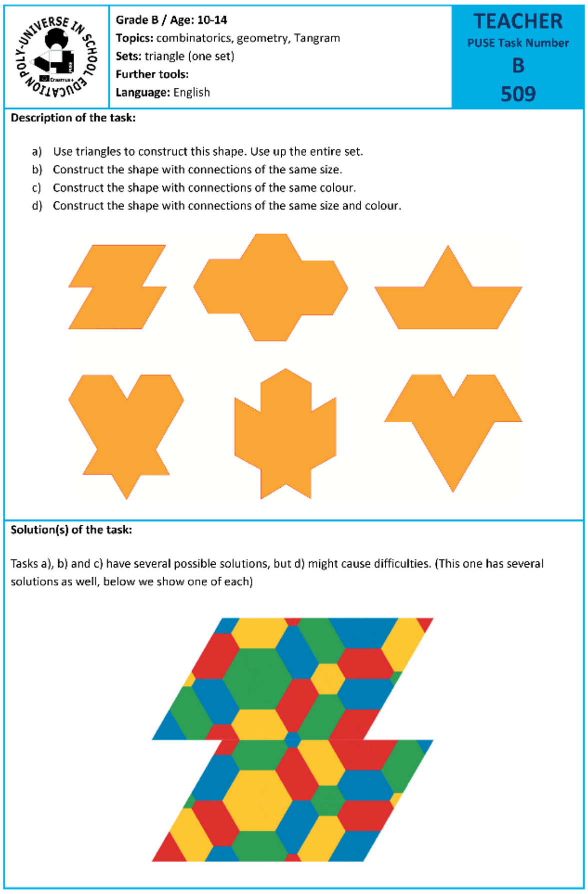
TEACHER
PUSE Task Number
B
212

Description of the task:

a) Construct the smallest possible regular hexagon with same colour and size connections. How many such hexagons can be built from a triangle set? (They don't have to be set out simultaneously; you can deconstruct them if you don't have enough elements. Two constructions are different if there is at least one element that has different neighbours in the constructions.)
b) Use the entire set to construct a regular hexagon with same colour and size connections. How many different constructions are possible this time?

Solution(s) of the task:

a) Due to the same colour connections, we will have a monochrome hexagon in the middle. There is only one way of constructing each of those central hexagons. Thus, for each colour and size, there is one solution, so we have 4 colours × 3 sizes = 12 different constructions. Some examples:
b) As seen in Task a) we can start building hexagons in 12 ways. But after we have our hexagon in Task a), there is only one way of continuing it as there is only one suitable element to be connected to each side of the triangle elements. (Two colours are defined by the connecting triangle elements, and the other two colours can only be changed among each other. The latter two can only be the triangle we are connecting with and the triangle we are connecting to.) By constructing them, we can check the fact that all 12 solutions can be finished. Some examples:



Grade B / Age: 10-14
Topics: combinatorics, geometry, Tangram
Sets: triangle (one set)
Further tools:
Language: English

TEACHER
PUSE Task Number
B
509

Description of the task:

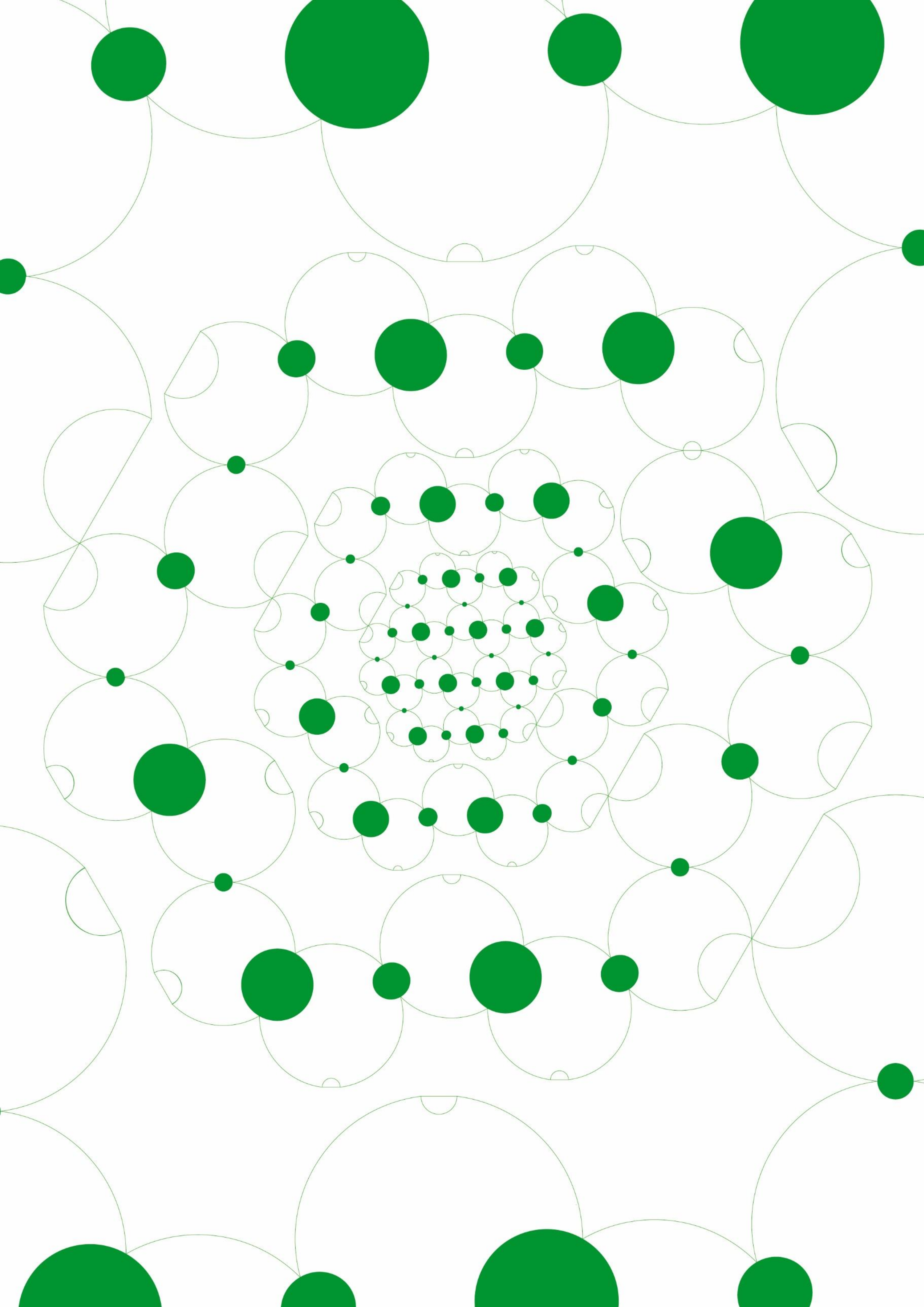
a) Use triangles to construct this shape. Use up the entire set.
b) Construct the shape with connections of the same size.
c) Construct the shape with connections of the same colour.
d) Construct the shape with connections of the same size and colour.

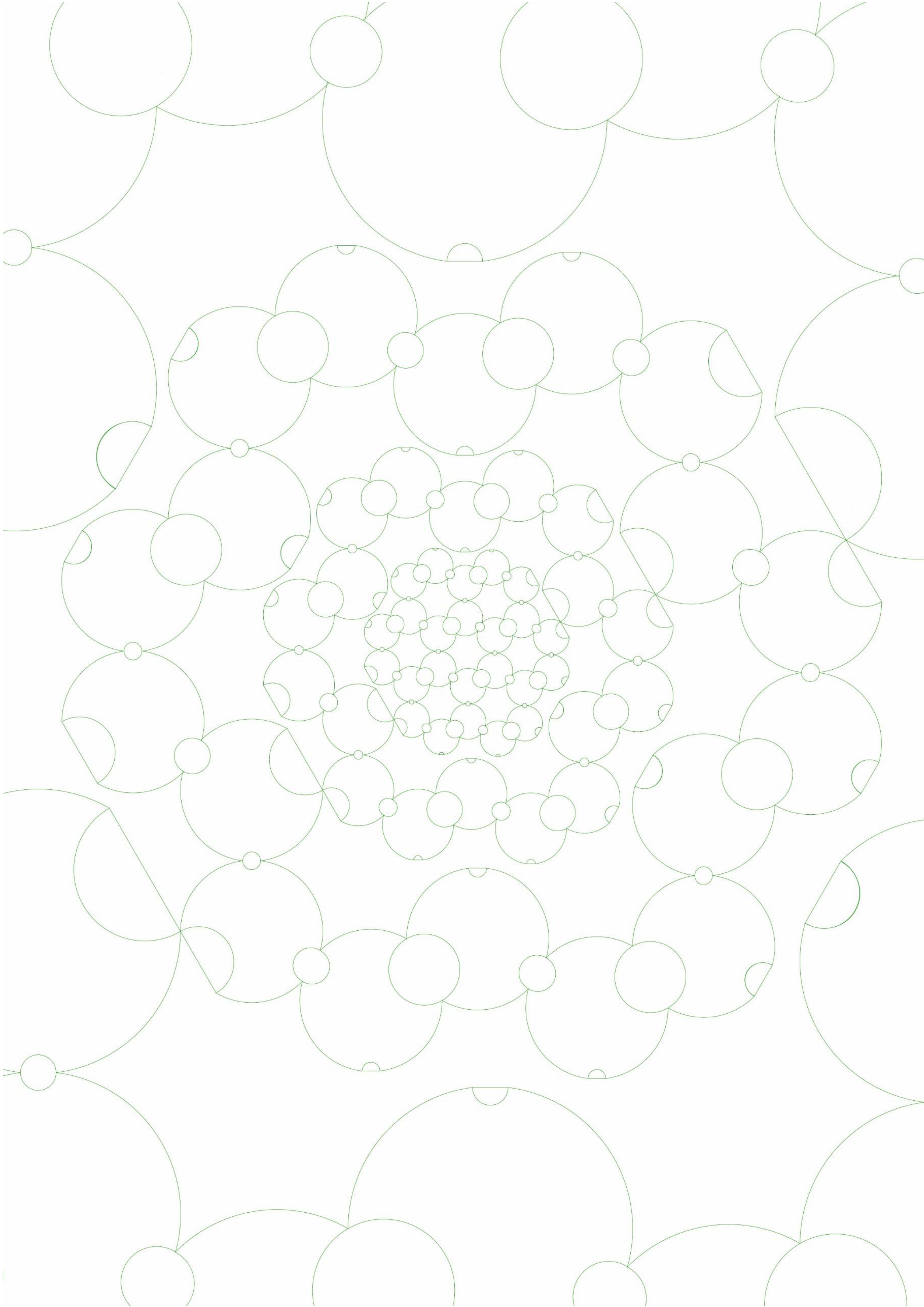
Solution(s) of the task:

Tasks a), b) and c) have several possible solutions, but d) might cause difficulties. (This one has several solutions as well, below we show one of each)

REFERENCE:

- PUSE (Poly-Universe in School Education) METHODOLOGY – Visual Experience Based Mathematics Education 2019; ISBN 978–615–81267–0–0; Edited by János Szász SAXON & Dr Eleonóra Stettner PhD; Published by Zsuzsa Dárdai, Poly-Universe Ltd, Szokolya, Hungary.
- PUSE Electronic version [PDF]; ISBN 978–615–81267–1–7: www.poly-universe.com
- PUSE E-learning trial version available: https://puse.education





V GOOD PRACTICES

In this part of the handbook are given 43 examples of using Poly-Universe materials in teaching in primary secondary school and university level. Teachers could use these examples in several ways:

a) Implement described examples as they are without modification in the classroom.
b) Teachers can modify and adapt these examples to their students and environment.
c) Teachers can use these examples as a starting point for developing their examples and approaches.

Examples of good practice are divided into examples of good practice in teaching mathematics, art and multidisciplinary. However, most examples of good practice can be applied in two or more subjects, so for example mathematics teachers can find inspiration in examples from the field of science, or vice versa. In the examples of good practice, suggestions are given for the age of the students, to which they can be adapted, teachers are advised to check the skills and abilities of their students, in order to adapt the teaching material to these facts, and thus follow the principles of constructivist learning. Certainly, the main goal of these examples is to give teachers an insight into the use of the Poly-Universe set in education at different levels, and different subjects and to encourage teacher's creativity for further application in the classroom.

5.1 Good practices — Teaching Mathematics

Good practice 1

The following task belongs to the topic of geometry-solid geometry, different cases of a net of a solid object. Used sets: square, triangle.

The next shape is a net of a solid object body, using 1 square and 4 triangles regardless of color.

a) Which solid object's net is shown in the following picture?

b) Find the total surface area and volume of the object if each edge of the base measures 8 cm!

The total surface area of a right regular square pyramid is made up of 1 square base, and 4 triangular faces/sides that are of equal size.

The area of the square base is a^2 where a is the length of the base:

$$T = a^2 = 64\ cm^2$$

Therefore, the equilateral triangles that make up the faces are congruent, so their heights are also congruent. The slant height m_a is the same on each face/side.

The slant height m_a can be calculated from the right triangle EFC. Since the point *F* is the midpoint of a base edge, $|FC| = \frac{1}{2}a$. The lateral edge length of the pyramid *b* is equal to the length of the base edge.

By Pythagoras' Theorem from right-triangle EFC, we have

$$m_a{}^2 = a^2 - \left(\frac{a}{2}\right)^2 = 64 - \frac{64}{4} = 48\ m_a = \sqrt{48} = 6{,}93\ cm$$

Area of 1 triangle face

$$T_1 = \frac{a \cdot m_a}{2}$$

and there's 4 of them.

Thus, the lateral area $= 4 \cdot \frac{a \cdot m_a}{2} = 2 \cdot a \cdot m_a$

Total surface area of a square pyramid:

$$F = T + 4 \cdot T_1 = 64 + 4 \cdot 6{,}93 = 91{,}72\ cm^2$$

The general volume of a pyramid formula is given as:

$$V = \frac{T \cdot m}{3}$$

T is the area of the square base, *m* is the height (height from the base to the apex).

The area of the square base is a^2 where *a* is the length of the base:

$$T = a^2 = 64\ cm^2$$

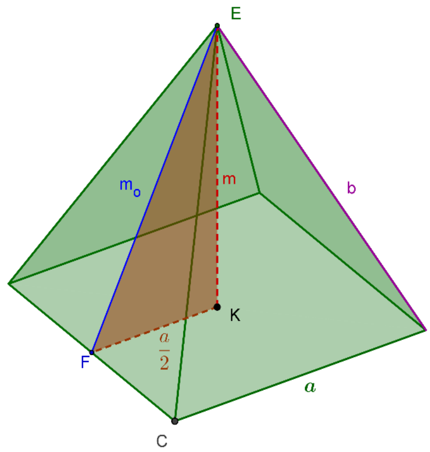
The height of the pyramid *m* can be calculated from the right triangle *EFK*. Since the point F is the midpoint of a base edge, |FK|=1/2 a.

$$m^2 = m_a{}^2 - \left(\frac{a}{2}\right)^2 = 48 - \frac{64}{4} = 32$$

$$m = \sqrt{32} = 5{,}66\ cm$$

The volume of the pyramid:

$$V = \frac{T \cdot m}{3} = \frac{64 \cdot 5{,}66}{3} = 120{,}75\ cm^3$$



- *Why this exercise is good:* Develops problem solving, logical thinking, inductive thinking, combinatorial thinking, and spatial vision.
- *Level of teacher training:* Secondary school
- *School subject(s):* Mathematics

Good practice 2

The task belongs to the topic of combinatorics. Used sets: square, triangle. The following picture shows a two-masted ship with 2 triangular sails on each mast. The different base colors of the sails mean different speeds. The sum of the speeds corresponding to the colors gives the final speed of the ship. Yellow moves the boat forward with a speed of 1 km/h, red with 2 km/h, green with 3 km/h and blue with 4 km/h.

a) How fast is the ship in the picture moving?

The left mast carries one red and one green base color sails, and the right mast carries one yellow and one blue base color sails. So exactly one piece of each base color. In this case the speed of the ship is 1+2+3+4 = 10 km/h.

a) Change the color of the sails so that the boat is traveling exactly with a speed of 13 km/h.

The essence of the task is to create a given sum using the numbers 1, 2, 3, 4.

There are several solutions to the problem and one of them is e.g. 3 green and 1 blue sails,

ie 3×3 + 1×4 = 13 km/h.

Note: Of course, the task can also be given so that students look for as many or all of the solutions as possible. It means a higher level of solutions if the order and position of the sails matter in the task – that is, which mast they are on, or below or above.

a) Change the color of the sails so that the boat is moving with the lowest possible speed.

The minimum speed corresponds to the yellow base color, so all sails should be yellow base color. Then the ship is moving at a speed of 4×1 = 4 km/h.

b) Change the color of the sails so that the boat is moving with the highest possible speed.

The maximum speed corresponds to the blue base color, so all sails should be blue base color. Then the ship is moving at a speed of 4×4 = 16 km/h.

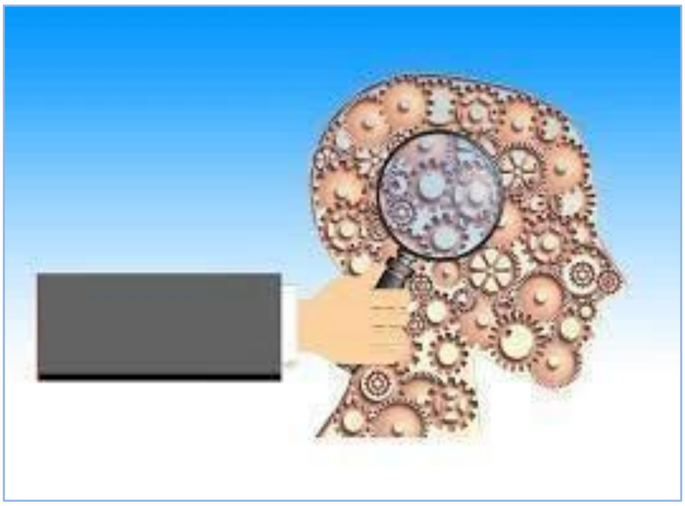
- *Why this exercise is good:* Develops problem solving, logical thinking, inductive thinking, and combinatorial thinking.
- *Level of teacher training:* Primary school upper grade, secondary school
- *School subject(s):* mathematics
- *Comments:* (any)

Good practice 3

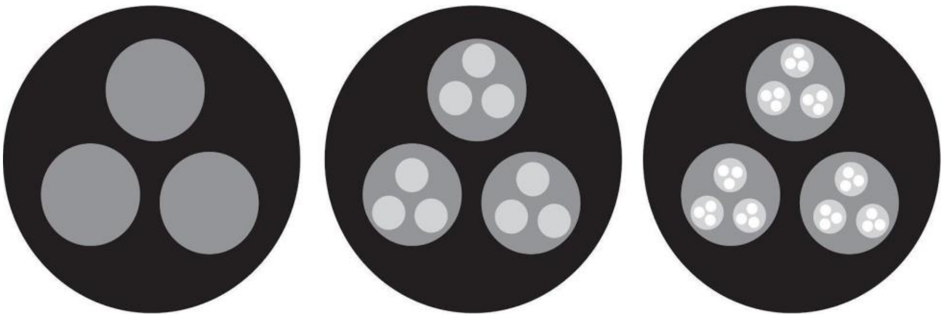
The relationship between the 'Polydimensional point' (defined in chapter2.5.1) and the human brain:

a. Humanity currently has a population of 8 billion, or 8×10^9
b. The number of human brain cells is about 100 billion, or 10^{11}

We attempt to relate the orders of magnitude between the didactic figure created during the creation of the 'Polydimensional point' and the Poly-Universe game, and the above numbers:



The way to perception the 'Poly-dimensional point' can be a simple logic experiment: If there is a set of planes made up by at least three other sets of planes that in turn include three further sets of planes each, and so forth ad infinitum, then we may witness the termination of the plane as a form, as it becomes a set of points. If, on the other hand, we take space, then the same process leads to the depletion of space or an object, and the substance, after reaching a density of infinite fineness, is immaterialized, is transformed in our mind definitively.



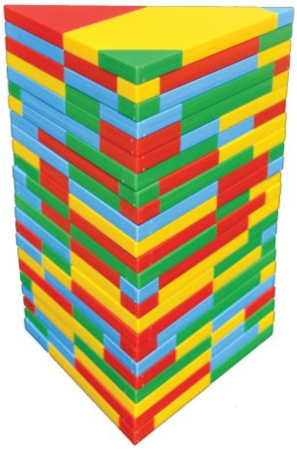
Questions:

1. How many depth layers within a 'Polydimensional point' must be reached before the number of points in the set equals the number of human inhabitants?
2. How many layers of depth within the 'Polydimensional point' must be the number of points in the set reach the number of average human brain cells?
3. How many brain cells in total do humans alive today have?
4. Could we give all the brain cells of all the people alive today a Poly-Universe game box, packed in different layouts?

5. How many people have lived on earth so far, if we include the number of people alive today?
   https://www.youtube.com/watch?v=PUwmA3Q0_OE&ab_channel=AmericanMuseumofNaturalHistory
6. Could we give every brain cell of every human being who has ever lived on earth a Poly-Universe game box, packed in different layouts?

- *Why this exercise is good:* This exercise is a true dimensional shift in thinking, helping us to find our real place in the Poly-Universe, the vertical web of microcosm and macrocosm...
- *Level of teacher training:* Subject teacher, secondary school
- *School subject(s):* Mathematics, biology, anthropology, informatics
- *Comments:* Combinatorial Packaging (PUSE Tasks 236C)

Based on the idea of the inventor, packaging happens as follows: each element of the 24-piece package is placed randomly above each other, and then covered with transparent foil. With this method, how many different ways are there to place the elements on each other?



Calculate the solution with triangle, square and circle base forms.

Solution(s) of the task:

Triangle: $6^{24} \times 24! \approx 2.9 \times 10^{42}$

Explanation: a triangle can be put down in 6 ways, so 24 triangles above each other give 6^{24} possibilities. We also need to calculate the different orders of putting the 24 elements on each other, the permutation of the 24 elements without repetition, which is 24!

Square: $8^{24} \times 24! \approx 2.9 \times 10^{45}$
Explanation: the only difference between the arrangement of squares and triangles is that the square can be put down in 8 ways. Other considerations are the same, see above.

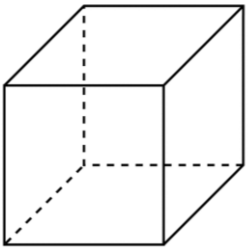
Circle: $24! \approx 6.2 \times 10^{23}$

Explanation: When packaging the circle set, semicircles of the same size need to be joined (so we do not calculate their rotated or reflected layouts). Thus, the formula of the cardinality of different stacks becomes a simple permutation without repetition.

Good practice 4

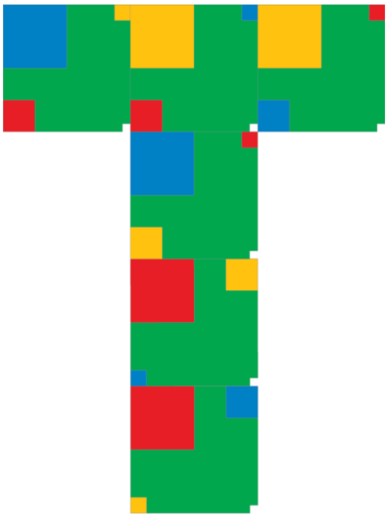
The following task belongs to the topic of geometry-solid geometry, different cases of a net of a solid object. Used sets: square.

Which object is in the next picture?

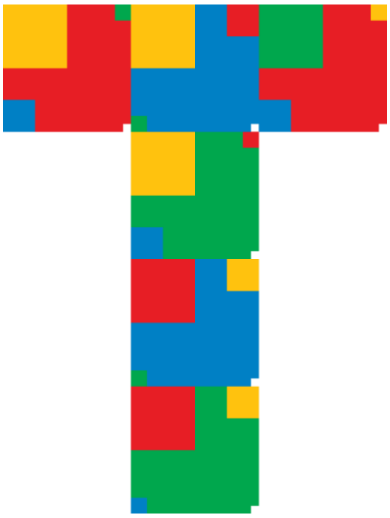


a) Create its net!

In the first part of the task, the net of the object can be created with elements of any color. E.g.



b) Create the net of the object (cube) so that the opposite surface faces are squares of the same base color and the adjacent side panels have a different base color.



- *Why this exercise is good:* Develops problem solving, logical thinking, inductive thinking, combinatorial thinking, spatial vision.
- *Level of teacher training:* Primary school upper grade, secondary school
- *School subject(s):* Mathematics
- *Comments: (any)*

Good practice 5

The task belongs to the topic of probability, partly a new task, partly a modification of the 510B PUSE task. Used sets: triangle, square.

Description of the task: construct a shape (target) from basic elements and calculate the probability of hitting a given color with a random shot.

We examine two cases:

I. Using the basic elements of a set we construct a closed shape, which will be the target. Students' task is to calculate the probability of hitting a given color on the target by random shooting.
II. We give the probability of a target's color and the students have to create a closed shape (target board) on which the probability of the given color corresponds to the given values.

Notes on the task:

a) The PUSE B510 task solves the second case in which the area of the fields of different colors are equal, i.e. their probability ratio is 1:1:1:1.

b) Both the triangle and square sets are suitable for constructing a target board. The probability of the colors will be the same if we use the same basic elements of triangles or squares. Because the ratio of the area of the small, medium, large form to the area of the base element is the same in triangle and square as well if we ignore the small square that is cut out from the corner of the base square.

Case I.

Using the basic elements of a set we construct a closed shape, which will be the target board. What is the probability that we will randomly hit a particular color on the target with a single shot?

a) Creating a target (closed shape) from a set of triangles:

– triangle of 1, 4, 9, 16 basic elements,
– hexagons of 6 or 24 triangles
– rhombus of 2, 8, 18 triangles
– trapezium, parallelogram,
– other shapes, symmetrical or not, e.g. a star of 12 triangles

Creating a target (closed shape) from a set of squares:

– square of 1, 4, 9, 16 basic elements,
– different size rectangles,
– symmetrical shapes, e.g a '+' or a cross shape
– other shapes.

b) Calculating probability:

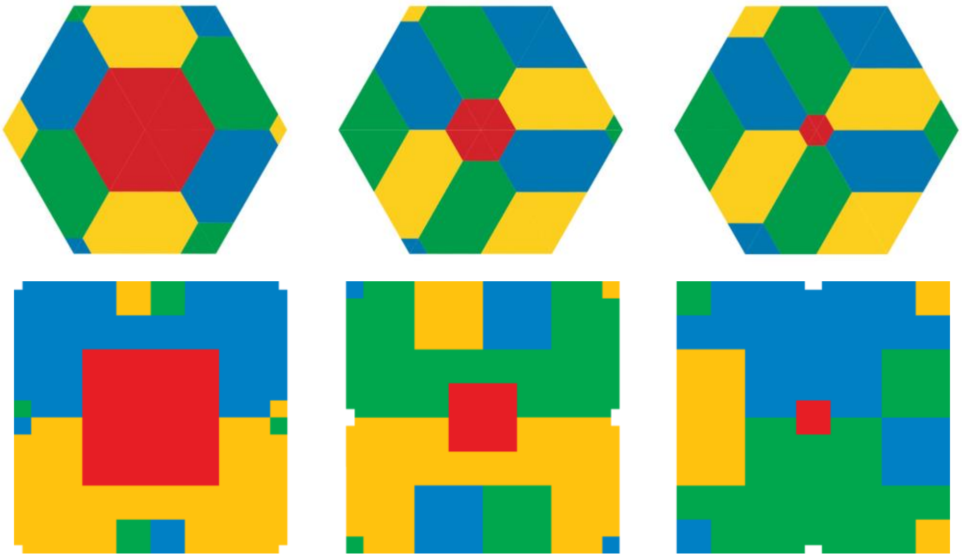
For both triangle and square basic elements (PUSE 133C task) the ratio of the area

– of the *large* form to the area of the basic element is 1/4,
– of the *medium* form to the area of the basic element is 1/16,
– of the *small* form to the area of the basic element is 1/64,
– of the *hexagon* form to the area of the basic element is 43/64,

For the calculation, let the area of the smallest field of the basic element be 1. Then the area of the small, medium and large field is 4, 16, 43 respectively, and the area of the basic element is 64. If we use n basic elements for the target board, the total area is $64n$.

The probability that we randomly hit a particular color on the target with a single shot is the ratio of the area with the given color to the total area of the target.

- Calculate the probability of hitting a selected color of the target (e.g. the center of the target).



- What is the probability that we will hit the field in the centre of the target (red color)?
- What is the probability that we will not hit the selected color?

The area of the total shape (target) is $64n$, where n is the number of basic elements used. The area of the field in the centre of the target (red color) for the hexagon target is $6 \cdot 16$, $6 \cdot 4$, $6 \cdot 1$ respectively and for the square target is $4 \cdot 16$, $4 \cdot 4$, $4 \cdot 1$. The probability that we will hit the red field in the centre is the ratio of the corresponding area to the total area, which is 1/4, 1/16, 1/64, respectively.

We know that the ratio of the area of the large, medium and small form to the area of the basic element is 1/4, 1/16, 1/64, respectively. This ratio is maintained in the total shape as well, since both the basic element and the form are used equally n times in the shape.

- What is the probability of hitting each of the four colors on the target with a single random shot?

To calculate the probability, create a table in which we write and summarize the size of the area of each colored field based according to the targets:

	base	large	medium	small	total
unit	43	16	4	1	64
1. color R	?	?	?	?	?
2. color Y					
3. color G					
4. color B					

The probability that we randomly hit a particular color on the target with a single shot is the ratio of the area with the given color to the total area of the target.

Case II.

We give the probability of a target's color and the students have to create a closed shape (target board) on which the probability of the given color corresponds to the given values.

- Only the probability of one color is given, the others are not specified. For which values of the probability can a target board be created?
- The probability of hitting the colors is given as the 1:1:1:1 ratio of the area of the colored fields. In this case we are looking for a shape in which the area of the fields of different colors are equal. How many basic elements of a set can be used to construct such a shape? For how many basic elements is it impossible to solve the task that the appropriate target board cannot be created? (PUSE 510B task).
- The probability of hitting the colors is generally given as the $a : b : c : d$ ratio of the area of the colored fields, according to which the target shape must be constructed. For which values and ratios $a : b : c : d$ has the task a solution?

To calculate the probability, we can create a table in which we write down and summarize the size of the area of each colored field.

Let us represent the number of elements according to their color and size with a 4×4 matrix **A**. Let the value of the colored fields on the basic element be a vector ***v*** **=** (43, 16, 4, 1) and the specified ratio is given as a vector ***a***. Then we look for a solution to the equation

$$A \cdot \vec{v} = \vec{a}$$

so that in the rows and columns of the matrix **A** the sum always gives the total number of basic elements.

For the ratio $a : b : c : d = 1 : 1 : 1 : 1$ we have different solutions, where the least number of basic elements is $n = 3$. For $n = 3$ a possible arrangement can be represented with the equation

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 43 \\ 16 \\ 4 \\ 1 \end{pmatrix} = 48 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

- *Why this exercise is good:* The task develops students' problem solving, logical thinking.
- *Level of teacher training:* According to the degree of difficulty of the task, it can be primary school upper grade, secondary school, teacher training
- *School subject(s):* Mathematics
- *Comments: (any)*

Good practice 6

The following task belongs to the topic of geometry-solid geometry, different cases of a net of a solid object. Used sets: square, triangle.

The next shape is a net of a solid object body, using 1 square and 4 triangles.

The next shape is a net of a pyramid, in the creation of which we can choose the base square. The base colors of the opposite triangular faces of the pyramid are the same. The base colors of adjacent triangular faces are different. The colors of the corner fields are not important for the connections.

How many different shapes (pyramids) can we create if we consider only the base color of the base square and triangular faces?

It is clear that if we swap two opposite triangles, it does not count as a new case-way (because only the base color of the triangles matter, and it does not change). Similarly, rotating the shape at an angle of 90 °, 180 °, or 270 ° also does not count as a new case-way.

Each case can be tabulated:

Base color of the base square	Triangle pairs	Number of ways
Yellow	2Y+2R, 2Y+2G, 2Y+2B, 2R+2G, 2R+2B, 2G+2B	6
Red	2Y+2R, 2Y+2G, 2Y+2B, 2R+2G, 2R+2B, 2G+2B	6
Green	2Y+2R, 2Y+2G, 2Y+2B, 2R+2G, 2R+2B, 2G+2B	6
Blue	2Y+2R, 2Y+2G, 2Y+2B, 2R+2G, 2R+2B, 2G+2B	6

For a given base square, we can create 6 different pyramid nets. If the base square can also be replaced, a total of 24 different pyramid nets can be made. The result can also be determined by calculation.

You can choose from 4 colors for the base color of the base square.

For the four triangles, the 2–2 pairs of triangles can be selected in a $\binom{4}{2}$ way due to the 4 base colors.

This is a total $4 \cdot \binom{4}{2} = 4 \cdot 6 = 24$ of: different shapes.

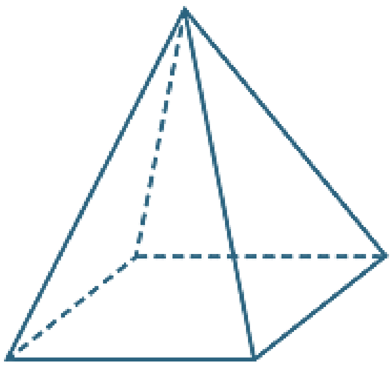
- *Why this exercise is good:* Develops problem solving, logical thinking, inductive thinking, combinatorial thinking, spatial vision.
- *Level of teacher training:* Primary school lower and upper grade, secondary school
- *School subject(s):* Mathematics
- *Comments: (any)*

Good practice 7

The following task belongs to the topic of geometry-solid geometry, different cases of a net of a solid object. Used sets: square, triangle.

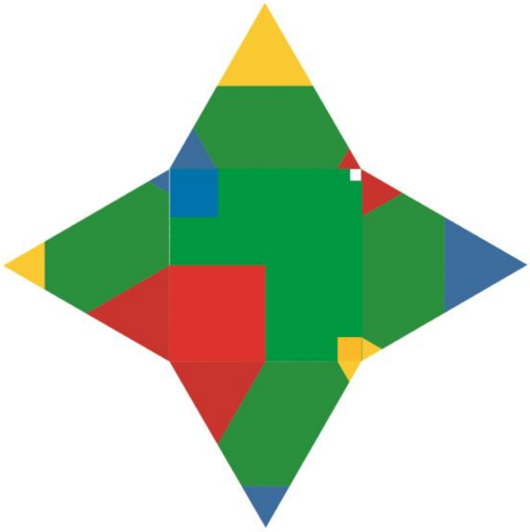
Which object is in the next picture?

a) Create its net!



In the first part of the task, the net of the object can be created with elements of any color. Shapes of the same size and color have to meet at the three vertices of the basic square.

b) Create the net of the object from/by elements of the same base color. Shapes of the same size and color have to meet at the three vertices of the basic square.



- *Why this exercise is good:* Develops problem solving, logical thinking, inductive thinking, combinatorial thinking, spatial vision.
- *Level of teacher training:* Primary school upper grade, secondary school
- *School subject(s*): Mathematics
- *Comments: (any)*

Good practice 8

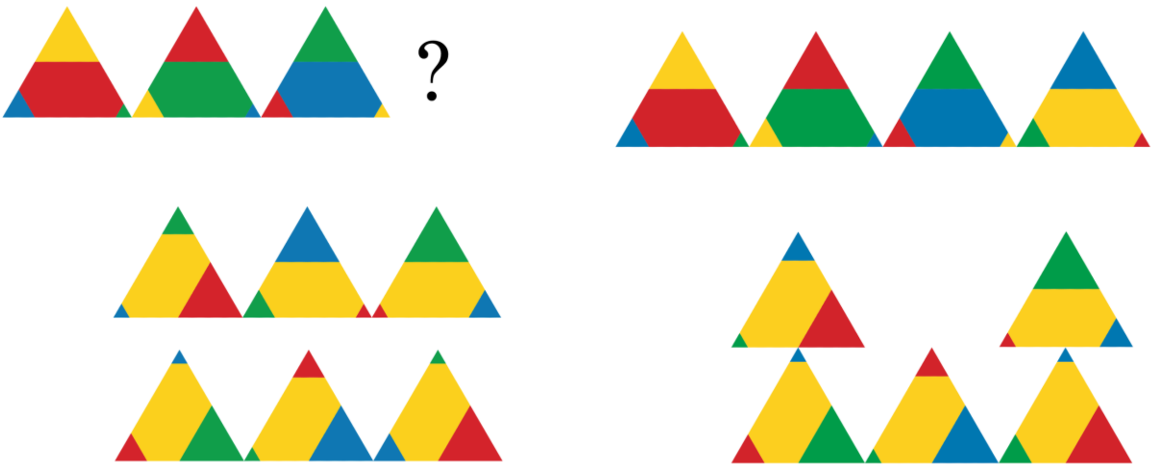
The following tasks belong to the topic of logic and logical thinking. There are different variations of a series of elements created according to a logical rule.

Task: recognize the rule and continue the series with one item/basic element.

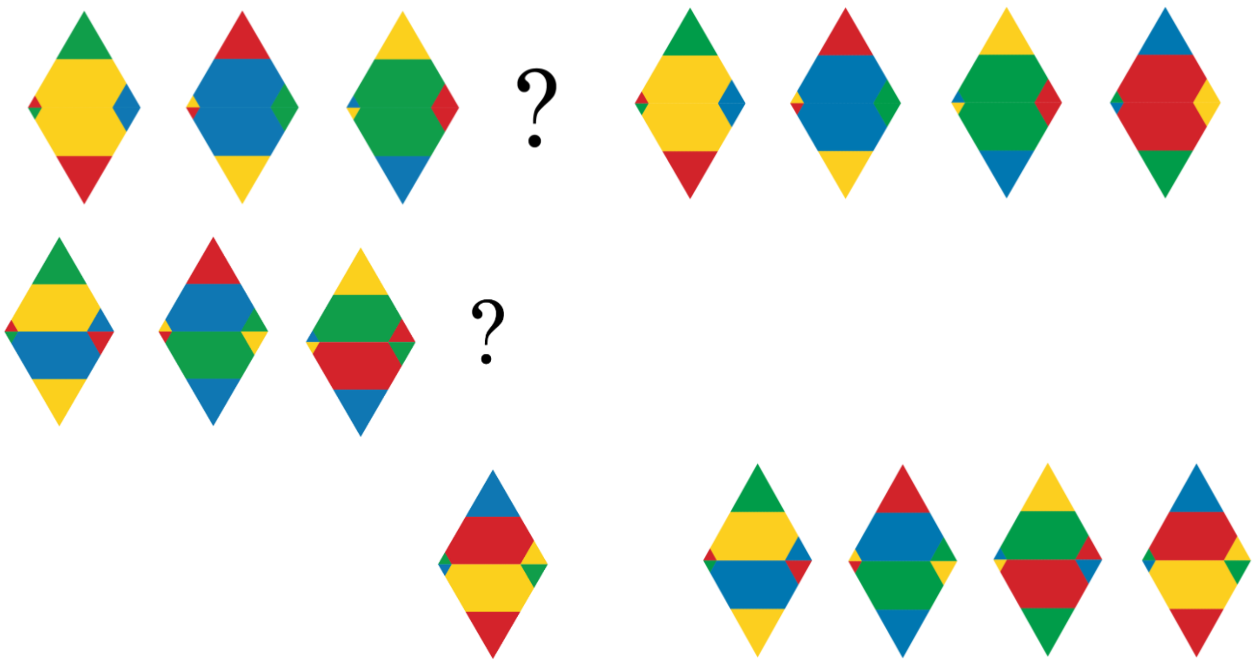
Continue the series of basic elements based on the rule observed. Explain by what rule you continued the series!

a) What will be the next item? Select it from the triangle basic elements.

b) What will be the next item? The missing item must be selected from the specified six basic triangle elements.



c) We have a series of shapes made up of two triangular elements. What will be the next item in the series? What rule did you apply to continue the series?



d) Work in pairs or small groups. One of the students makes a task in which he creates a series of elements according to a rule. The others shall find the rule and continue the series.

Depending on the task, the series can be continued:

– with only one element fulfilling the conditions of the rule,
– with several different elements, following the observed rule or rules,
– with no matching element.

- *Why this exercise is good:* The task develops problem solving, logical thinking, and inductive thinking. Developing creativity: students work in pairs or groups to create sequences of elements for each other.
- *Level of teacher training:* Primary school lower grade (6–10 years)
- *School subject(s):* Mathematics
- *Comments:* The series can be created of square and circle basic elements in an analogous way.

Good practice 9

The following tasks belong to the topic of logic and logical thinking. There are different variations of a series of elements created according to a logical rule.

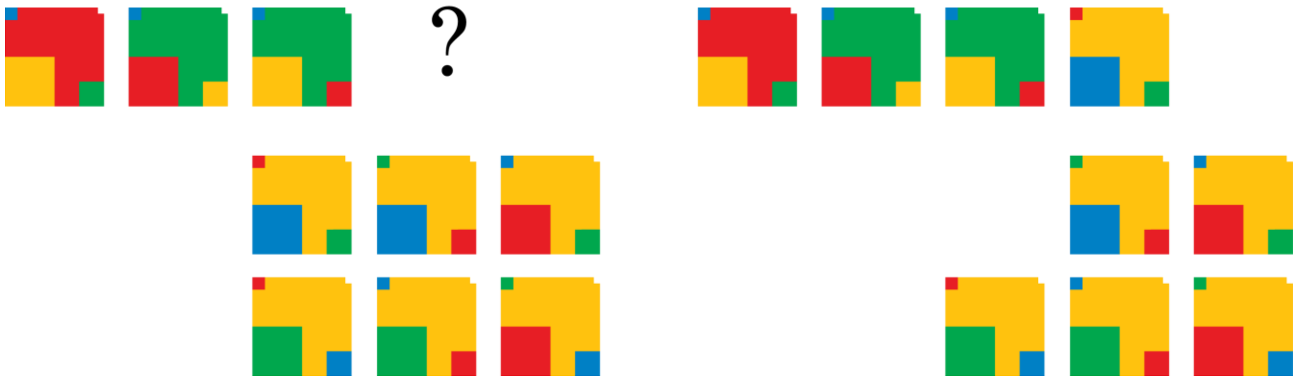
Task: recognize the rule and continue the series with one item/basic element.

Continue the series of basic elements based on the rule observed. Explain by what rule you continued the series!

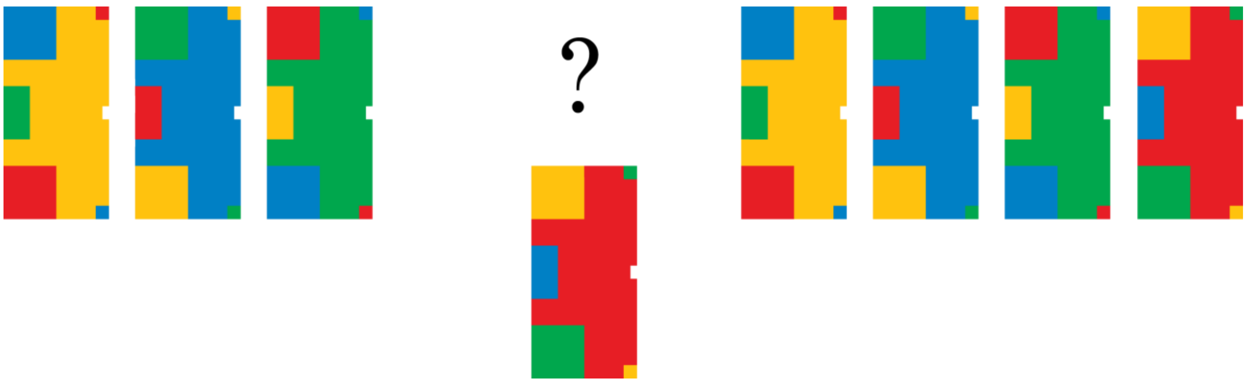
a) What will be the next item? Select it from the square basic elements

What will be the next item?

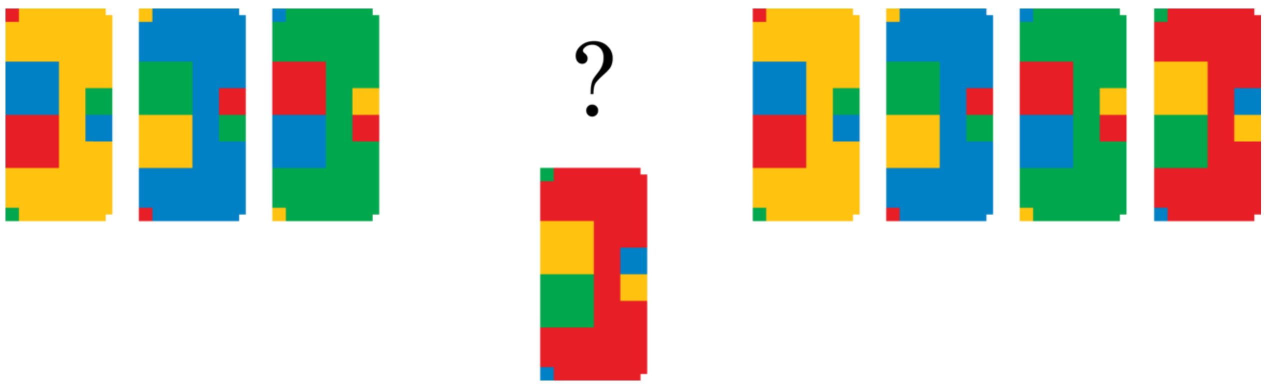
b) What will be the next item? The missing item must be selected from the specified six basic elements.



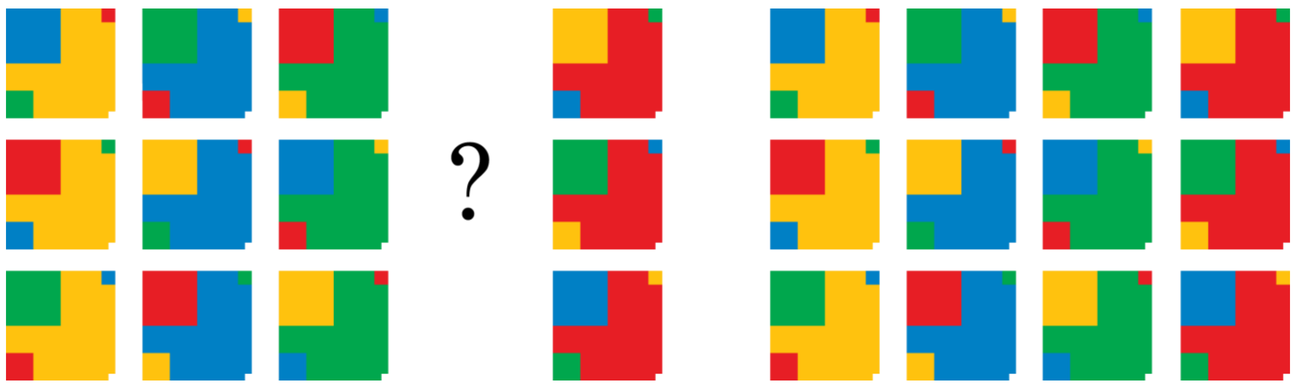
c) We have a series of shapes made up of two square basic elements. What will be the next item in the series? What rule did you apply to continue the series?



What will be the next item?



d) Continue the series of columns.



e) Work in pairs or small groups. One of the students makes a task in which he creates a series of elements according to a rule. The others shall find the rule and continue the series.

Depending on the task, the series can be continued:

– with only one element fulfilling the conditions of the rule,
– with several different elements, following the observed rule or rules,
– with no matching element.

- *Why this exercise is good:* The task develops problem solving, logical thinking, and inductive thinking. Developing creativity: students work in pairs or groups to create sequences of elements for each other.
- *Level of teacher training:* Primary school lower grade (6–10 years)
- *School subject(s)*: Mathematics
- *Comments:* Triangle and circle basic elements can be used for creating tasks in an analogous way.

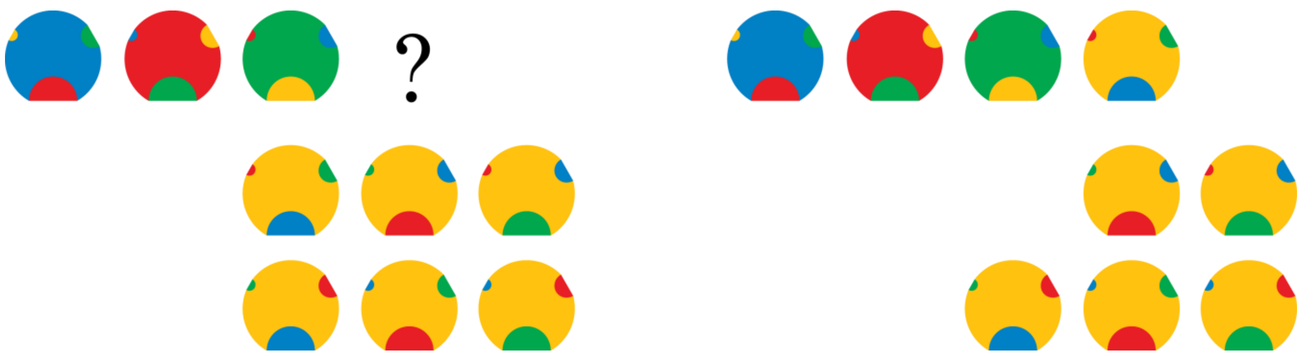
Good practice 10

The following tasks belong to the topic of logic and logical thinking. There are different variations of a series of elements created according to a logical rule.

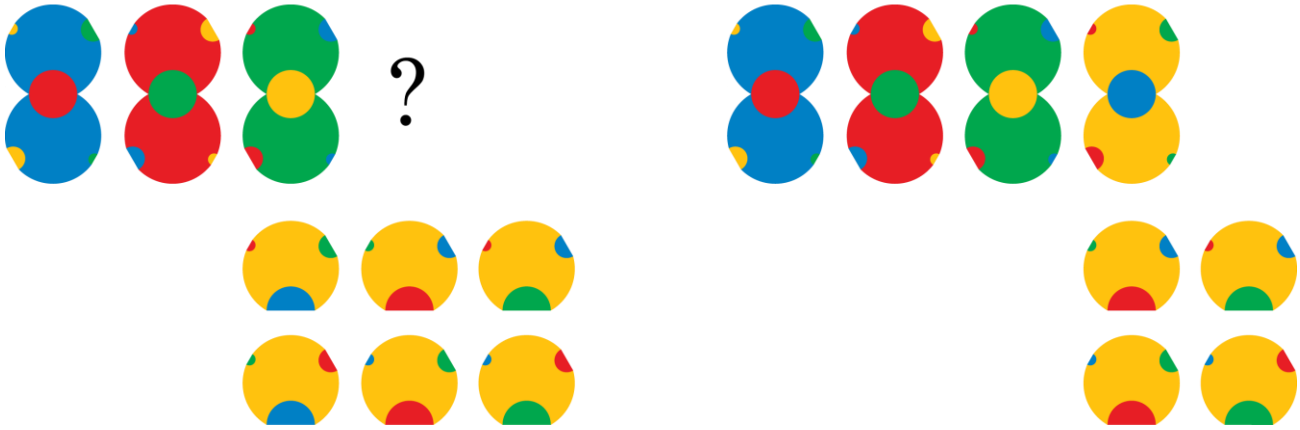
Task: recognize the rule and continue the series with one item/basic element.

Continue the series of basic elements based on the rule observed. Explain by what rule did you continue the series!

a) What will be the next item? The missing item must be selected from the specified six basic elements

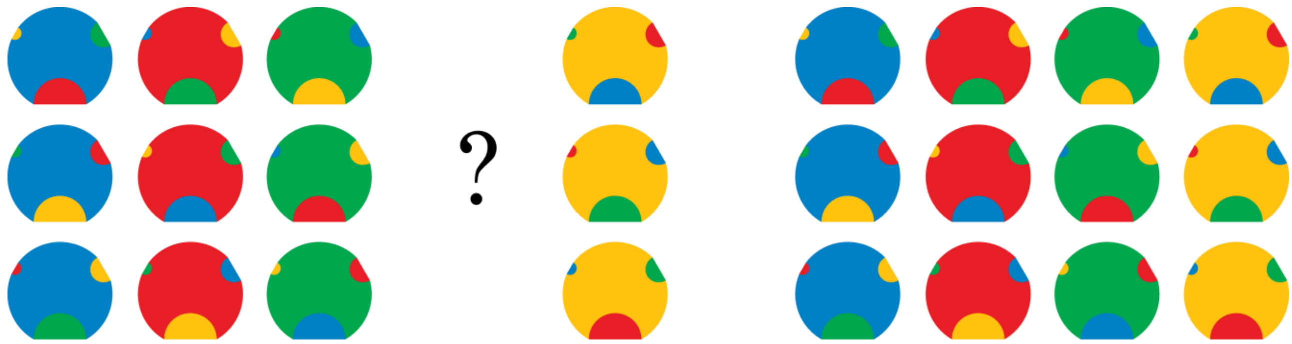


What will be the next item?



b) What will be the next item? What rule did you apply to continue the series?

c) Continue the series of columns.



d) Work in pairs or small groups. One of the students makes a task in which he creates a series of elements according to a rule. The others shall find the rule and continue the series.

Depending on the task, the series can be continued:

– with only one element fulfilling the conditions of the rule,
– with several different elements, following the observed rule or rules,
– with no matching element.

- *Why this exercise is good:* The task develops problem solving, logical thinking, and inductive thinking. Developing creativity: students work in pairs or groups to create sequences of elements for each other.
- *Level of teacher training:* Primary school lower grade (6–10 years)
- *School subject(s):* Mathematics
- *Comments:* Triangle and square basic elements can be used for creating tasks in an analogous way.

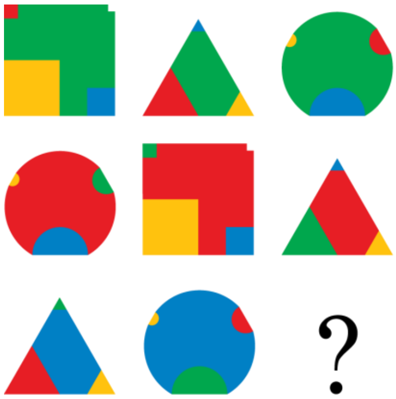
Good practice 11

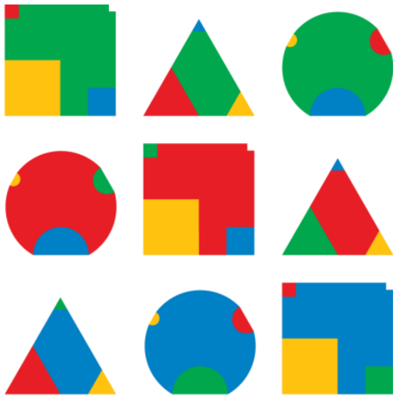
The following tasks belong to the topic of logic and logical thinking. There are different variations of a series of elements created according to a logical rule.

Basic elements used: triangle, circle, square.

Task: recognize the rule and continue the series with one item/basic element.

Complete the 3×3 layout below with the missing item.





The essence of the task:

– every basic element in each rows has different form
– every basic element in each rows has the same base color
– large, medium, small forms have different color in the same rows

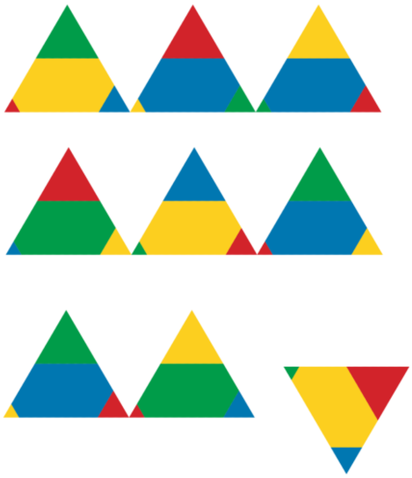
- *Why this exercise is good:* The task develops problem solving, logical thinking, and inductive thinking. Developing creativity: students work in pairs or groups to create sequences of elements for each other.
- *Level of teacher training:* Primary school lower and upper grade
- *School subject(s):* Mathematics
- *Comments: (any)*

Good practice 12

The following tasks belong to the topic of logic and logical thinking. There are different variations of a series of elements created according to a logical rule.

Basic elements used: triangle.

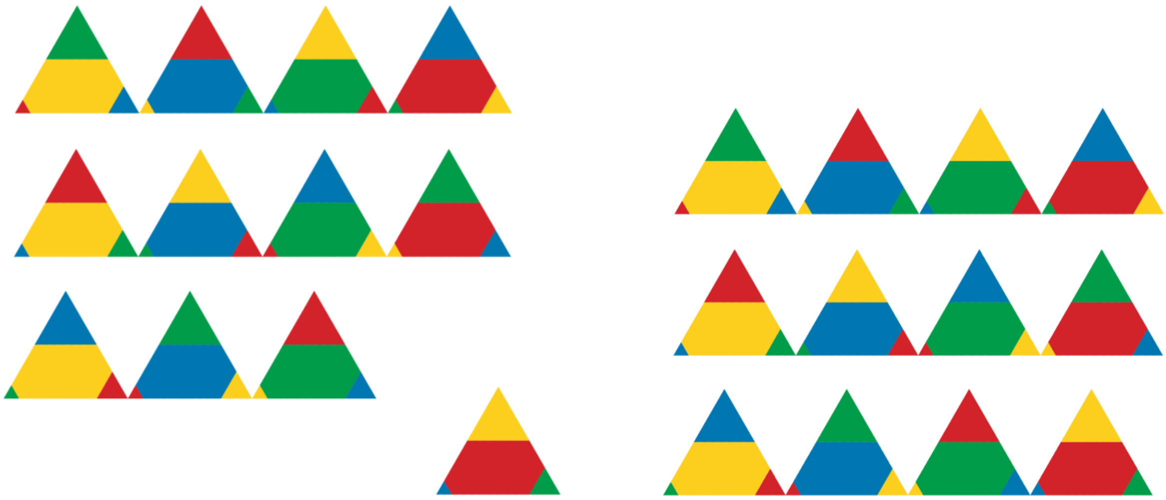
a) Using any of the 9 basic elements in the triangle set create a 3×3 layout where in each row and each column every form's size is in a different color.



The layout cannot be solved under the given conditions.

b) Using any of the 16 basic elements in the triangle set create a 4×4 layout where in each row and each column every form's size is in a different color.

c) Complete the 3×4 layout below with the missing item.



- *Why this exercise is good:* The task develops students' problem solving, logical thinking, and inductive thinking.
- *Level of teacher training:* Primary school lower and upper grade
- *School subject(s):* Mathematics
- *Comments: (any)*

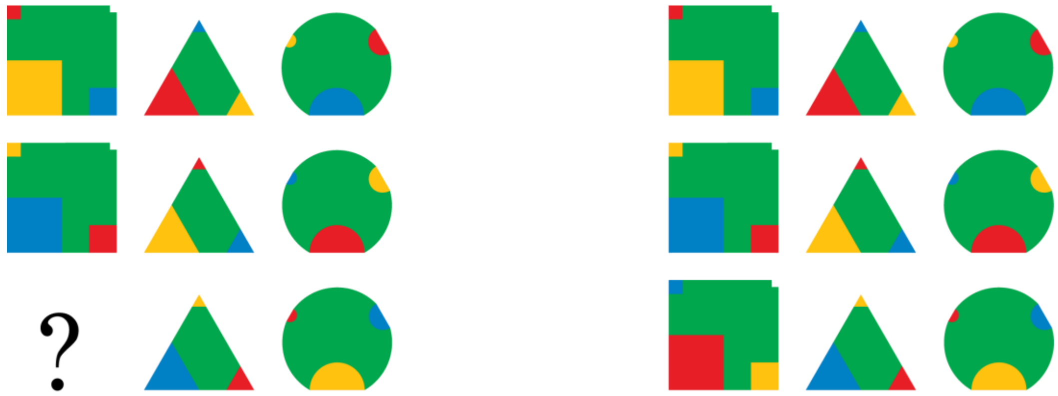
Good practice 13

The following tasks belong to the topic of logic and logical thinking. There are different variations of a series of elements created according to a logical rule.

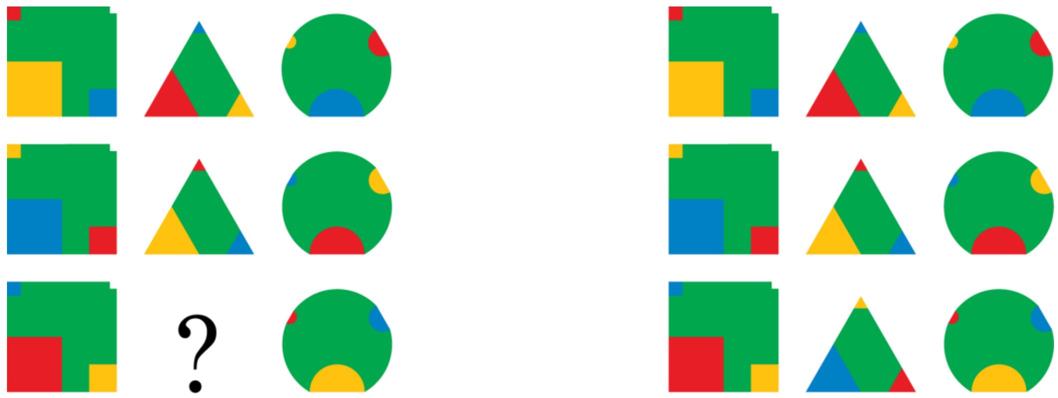
Basic elements used: triangle, circle, square.

Task: recognize the rule and complete the 3×3 layout below with the missing item.

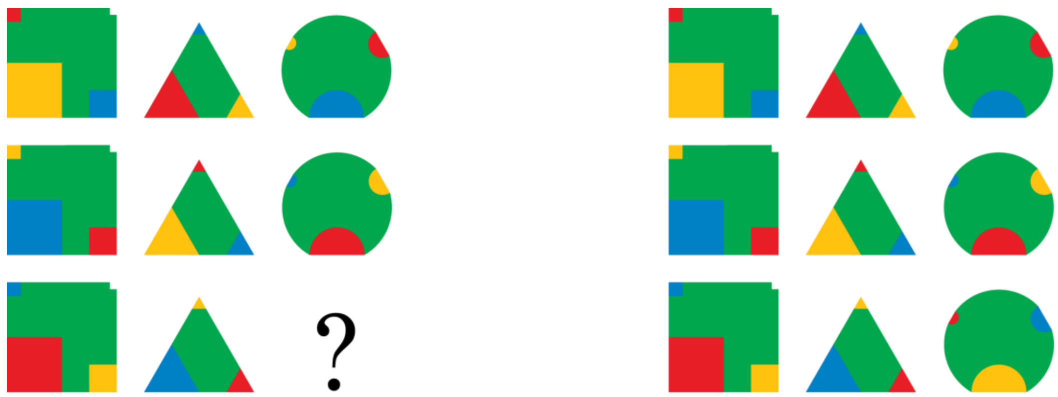
a)



b)



c)



The essence of the task:

– every basic element in each rows has different form
– every basic element in each rows has the same base color
– large, medium, small forms have different color in the same rows

- *Why this exercise is good:* The task develops problem solving, logical thinking, and inductive thinking. Developing creativity: students work in pairs or groups to create sequences of elements for each other.
- *Level of teacher training:* Primary school lower and upper grade
- *School subject(s):* Mathematics
- *Comments: (any)*

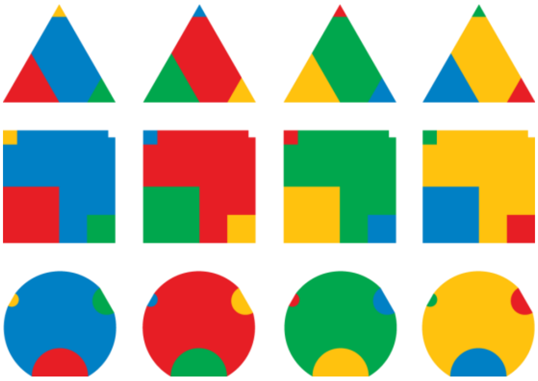
Good practice 14

The following tasks belong to the topic of logic and logical thinking. There are different variations of a series of elements created according to a logical rule.

Basic elements used: triangle. circle, square.

a) Using 4 basic elements from each of the three sets create three rows of a series of elements following the same rule in each row.

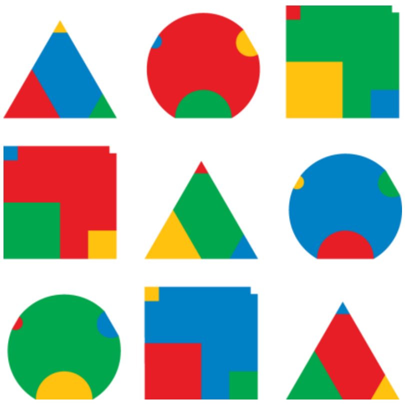
Possible solution:



b) Using 3 basic elements from each of the three sets create 3×3 layout where in each row and each column:

– every basic element has different form
– every basic element has different base color
– large, medium, small forms have different color

Possible solution:



- *Why this exercise is good:* The task develops students' problem solving, logical thinking, and inductive thinking.
- *Level of teacher training:* Primary school lower and upper grade
- *School subject(s):* Mathematics
- *Comments: (any)*

Good practice 15

The following task belongs to the topic of geometry-calculation of area. Used sets: square, triangle, and circle.

There is a bird on the following picture

a) Determine how many pieces of each element were used to create the bird.

To create the bird, 2 squares, 9 triangles, and a circle were used.

b) Find the area of the shape, if the length of the side of the square is 8 cm!

The area of each element.

Square: $T_1 = a^2 = 64\ cm^2$, where *a* is the length of the side of the square.

Triangle: $T_2 = \frac{a \cdot m_a}{2}$, where *a* is the length of the side of the equilateral triangle, and m_a is the height of the triangle.

We can calculate the height of the triangle using Pythagoras' theorem:

$$m_a = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \sqrt{64 - 16} = \sqrt{48} = 6{,}93\ cm.$$

The area of the triangle: $T_2 = \frac{a \cdot m_a}{2} = \frac{8 \cdot 6{,}93}{2} = 27{,}72\ cm^2$.

The diameter of the circle symbolizing the bird's head is equal to the side length of the square, from which the radius is *r = 4 cm*.

The area of the circle: $T_3 = \pi \cdot r^2 = \pi \cdot 16 = 50{,}24\ cm^2$.

Area of the shape (bird):

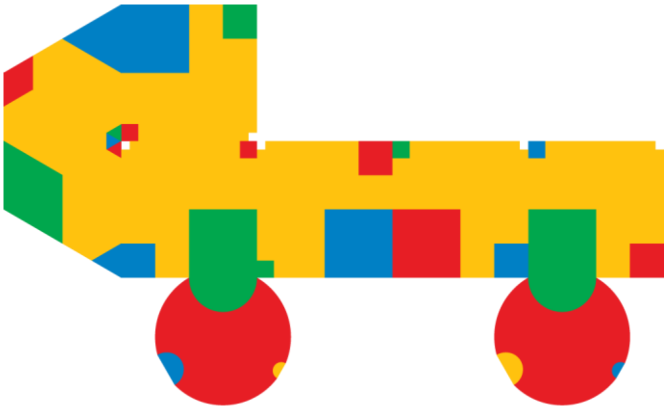
$$T = 2 \cdot T_1 + 9 \cdot T_2 + T_3 = 2 \cdot 64 + 9 \cdot 27{,}72 + 50{,}24 = 427{,}72\ cm^2$$

- *Why this exercise is good:* Develops problem solving, logical thinking, inductive thinking, combinatorial thinking.
- *Level of teacher training:* Primary school upper grade, secondary school
- *School subject(s):* Mathematics
- *Comments: (any)*

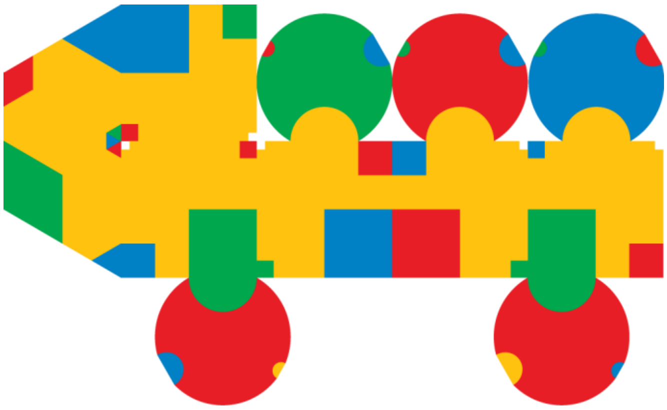
Good practice 16

The task belongs to the topic of combinatorics. Used sets: square, triangle, and circle.

The following picture shows a truck. Create it using the square, triangle and circle.



a) The truck's platform can hold a total of 3 circle consignments. How many ways can they load 3 consignments onto the truck’s platform if their order matters and they have to have a different base color?



For consignments, we can choose from 4 basic colors to 3 places so that the colors cannot be repeated and the order of colors matters.

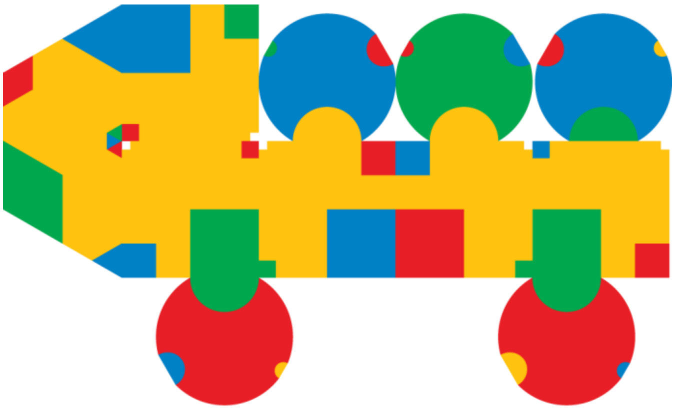
Behind the driver's cab, you can choose from 4 colors for the first place, 3 colors for the second place and 2 colors for the third place. This is a total of: $4 \cdot 2 \cdot 3 = 24$ different ways.

a) Different base colors for circle consignments mean different weights. The sum of the weights corresponding to the colors gives the total weight of the truck's load. Yellow corresponds to a mass of 4 tonnes, red to 1 tonne, green to 2 tonnes and blue to 3 tonnes.

What is the total mass of the consignments shown in the previous picture?

There are 1 green, 1 red and 1 blue consignment on the truck platform. This is a total of 2 + 1 + 3 = 6 tonnes.

b) The maximum load capacity of the truck is 8 tonnes. In how many ways can 3 consignments be loaded onto the truck's platform to have a total mass of exactly 8 tons?



The essence of the task is to create a given sum using the numbers 1, 2, 3, 4.

There are several solutions to the problem and one of them is e.g. 1 green and 2 blue consignments, i.e. $1 \cdot 2 + 2 \cdot 3 = 8$ t.

Note: Of course, the task can also be given so that students look for as many or all of the solutions as possible. It means a higher level of solutions if the order and position of the consignments matter in the task.

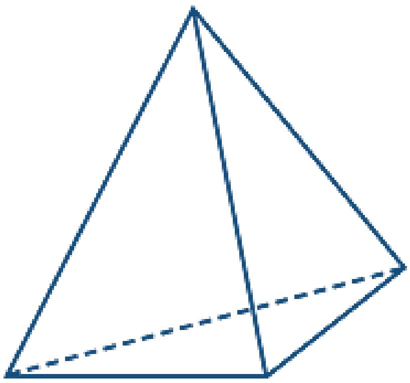
- *Why this exercise is good:* Develops problem solving, logical thinking, inductive thinking, combinatorial thinking.
- *Level of teacher training:* Primary school upper grade, secondary school
- *School subject(s):* Mathematics
- *Comments: (any)*

Good practice 17

The following task belongs to the topic of geometry-solid geometry, different cases of a net of a solid object. Used sets: triangle.

Which object is in the next picture?

a) Create its net!



In the first part of the task, the net of the object can be created with elements of any base color.

b) Create the net of the object from/by elements of the same base color.

- *Why this exercise is good:* Develops problem solving, logical thinking, inductive thinking, combinatorial thinking, and spatial vision.
- *Level of teacher training:* Primary school upper grade, secondary school
- *School subject(s):* Mathematics
- *Comments: (any)*

Good practice 18-19

Construct a Poly-Universe triangle and square element in GeoGebra.

PUSE Task Number: 519 BC, 520 BC

Further development of the tasks: Do not fix the ratio, but change it dynamically with the slider. Let us also allow the negative ratio. In this case, the smaller elements are outside the base element. Which is the ratio at which the large and medium triangles (squares) just touch each other? (Golden ratio) Why?

https://www.geogebra.org/classic/whxahtat

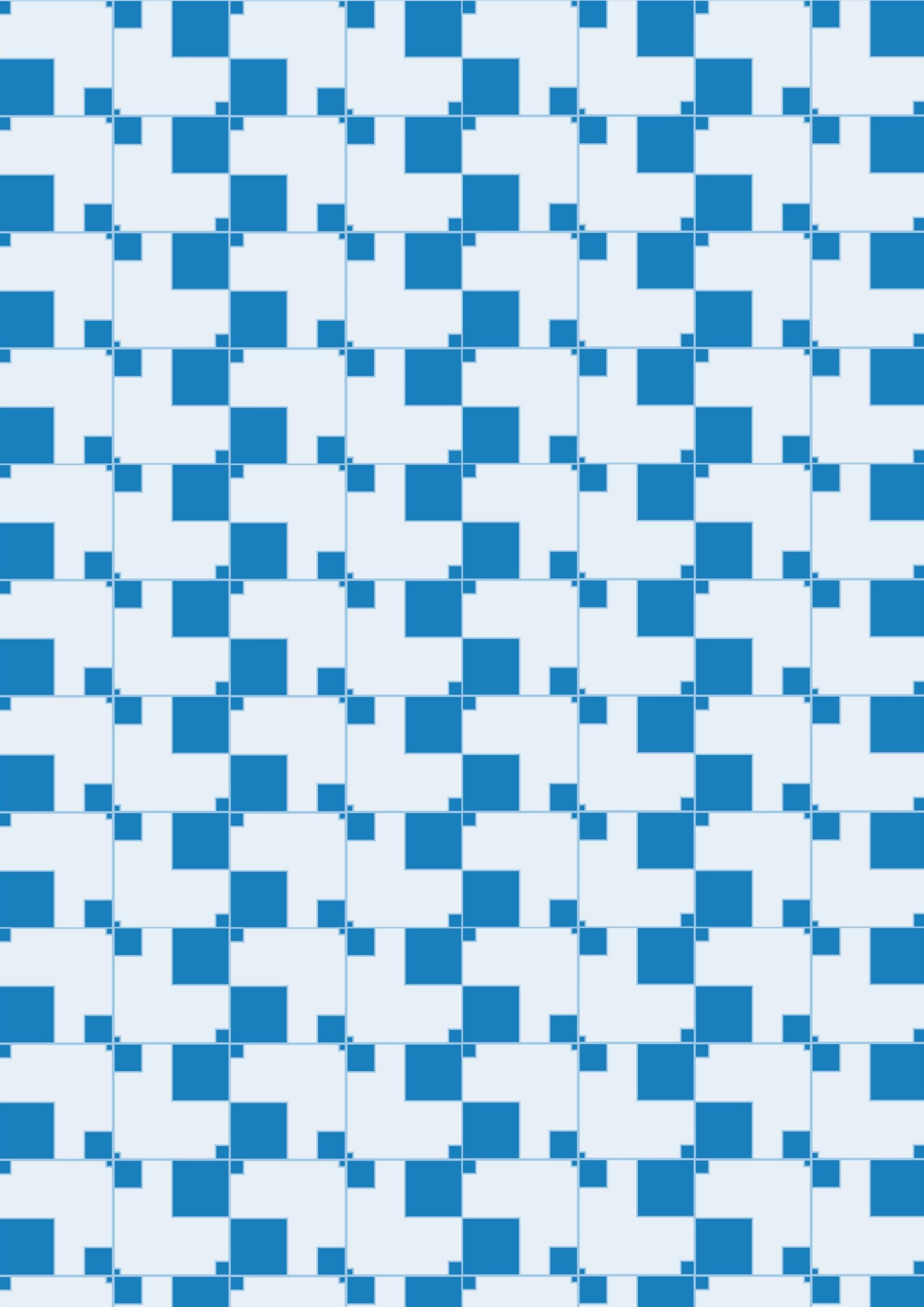
Slider min: –1, max: 1

https://www.geogebra.org/classic/zkncndfk

Slider min: –2, max: 2, translucent colors so that overlapping cases are more visible

You can also perform similar tasks with the square element.

- *Why this exercise is good:* It shows well the possibilities of expanding and varying the set. János Szász Saxon also made other works with different proportions from the ½ ratio, also those where the smaller shapes fall in the outer region of the basic element. Let's find and study them! Mathematics: properties of central similarity, (dilation) at different ratios (at different dilations factor), > 1, < 1, negative ratio can be dynamically illustrated. The task also provides an opportunity to talk about the golden ratio. Multidisciplinary approach; mathematics, art, informatics
- *Level of teacher training:* Elementary, subject teacher, secondary school, etc.
- *School subject(s):* Mathematics, IT, Arts
- *Comments:* The tasks are not fully developed everywhere, they only give ideas and encourage the teachers and students further experimentation and development.



5.2 Good practices — Teaching Art

Good practice 20

A century ago, the Russian constructivist painter Kasimir Malevich formed the creation of the black square on a white background, and other suprematism basic elements, which are: CIRCLE–SQUARE–CROSS (Figure 1). The opposite pair of a square is the cross because if we put the picture fields together, the cross divides the square into four parts, that is, it tries to break it down.

Figure 1

Malevich had the greatest intellectual influence on the work of Saxon, the inventor of the Poly-Universe, and he liked to use these elements in his painting. Of course, he inserted them into his polydimensional imaging system, as shown below, and writes about it (Figure 2).

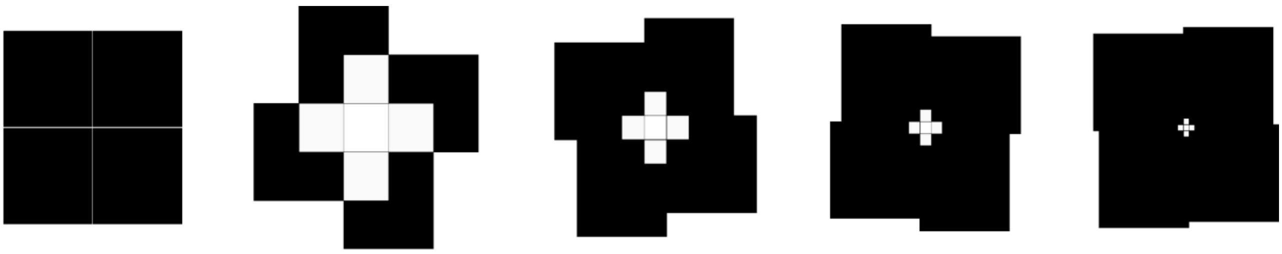


Figure 2:

'I did my next experiment during the 'supreMADIsm' (www.mobilemadimuseum.hu) festival organized in 2006 in Moscow. I embedded the white cross, one of Malevich's basic suprematism elements into the other basic suprematism element, the black square, the former trying to deconstruct the latter. The confrontation of these two forms can be found in my earlier works of art, but in the present case transcending the geometric shaping did not take place in terms of some 'Russian spiritualism', but rather pragmatically. Before that we had been able to understand the scientific nature of my works in their fractal character described by the 'dimension shifting'. Now, the main field of interest of mine included dividing the plane surfaces with the help of geometric figures and rearranging Malevich's cross in a poly-dimensional way. During this work I created horizontal and diagonal constructions, poly-dimensional cross-icons, but in this case, as a result of the closed system of the form, there arose finite, only about a dozen of variations for each. Strict monochrome, or more unambiguously, the black and white contrasts produce a powerful psycho-logical effect besides the variations of visual logical structures.' (Figure 3-4)

We now show a work with a horizontal and a diagonal arrangement.

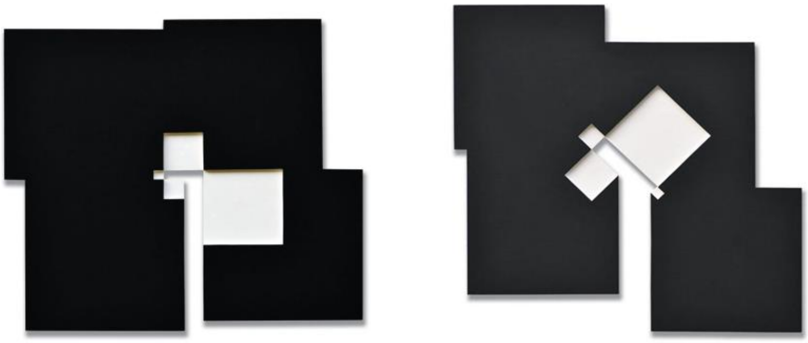


Figure 3

Working in small groups, cut out the basic forms from black and white paper and put together the works above. Then cut out as many basic elements as possible and find all the possible layouts and organize a comparative exhibition of them. How many such works can be made?

- *Why this exercise is good:* It presents art in an interesting way.
- *Level of teacher training:* Subject teacher, secondary school
- *School subject(s):* Art history, creative arts, combinatory, constructive play
- *Comments:* We show all the solutions that should have been made (Figure 4).

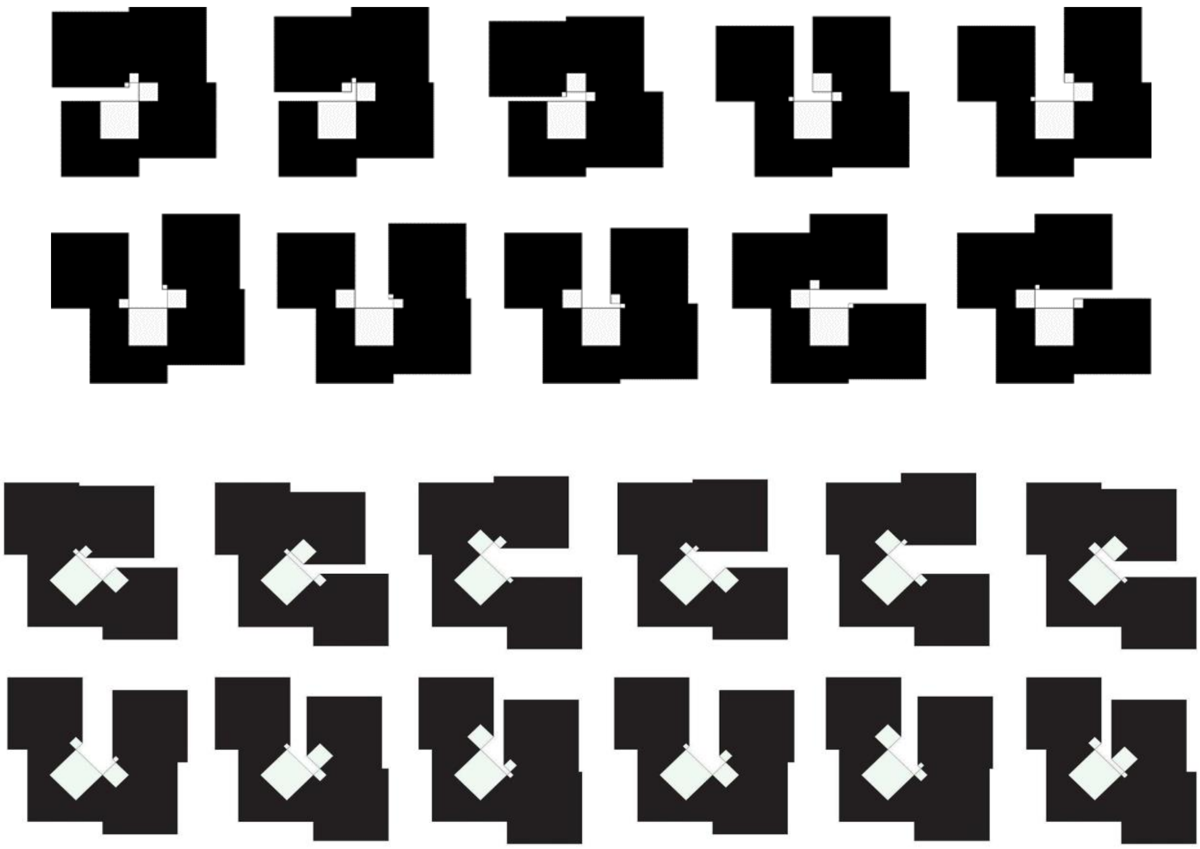
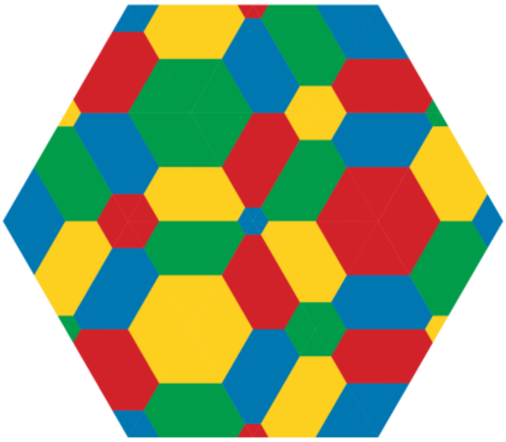
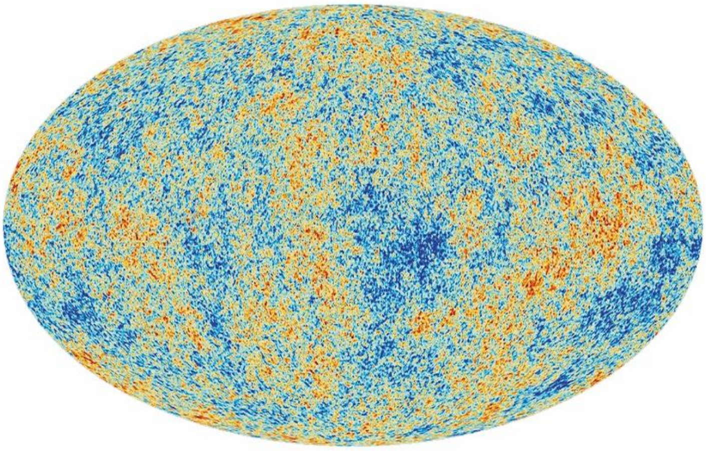


Figure 4

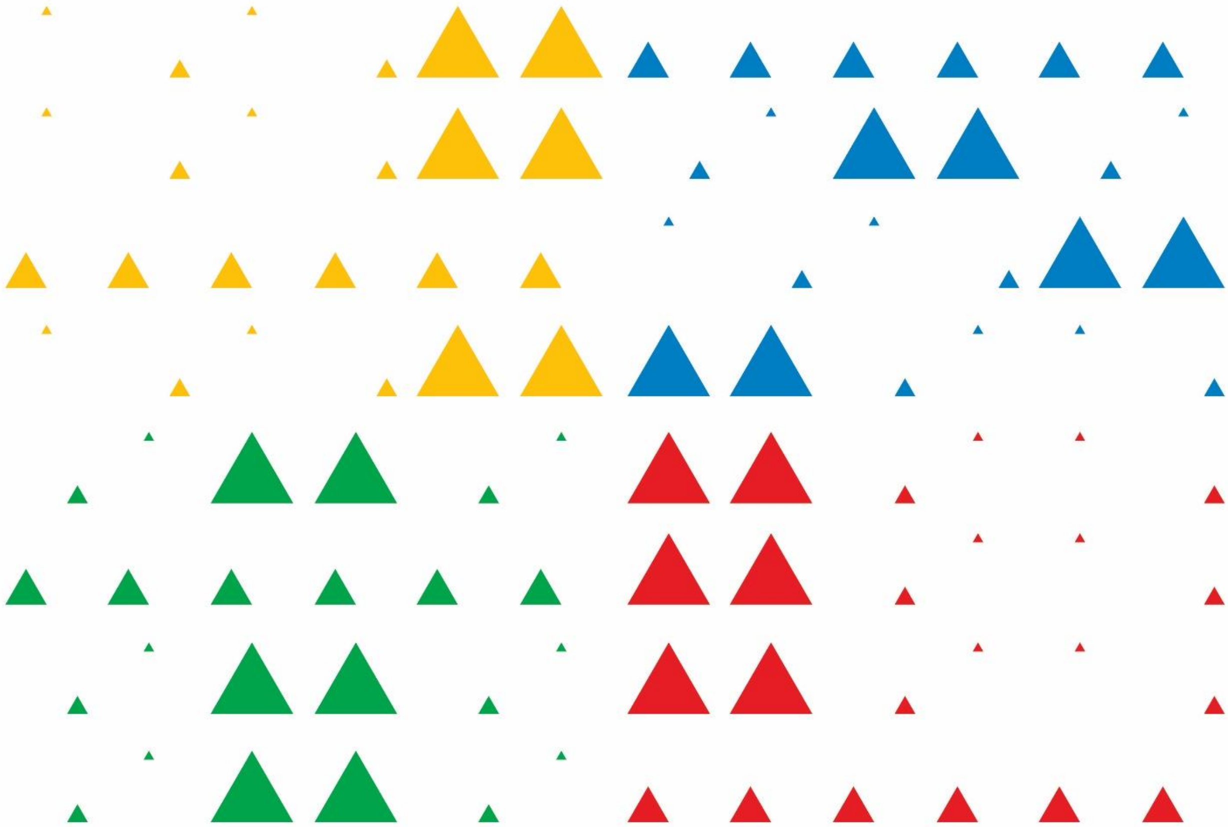
Good practice 21

Inserting the elements of the Poly-Universe into real cosmic space. A new approach to building basic elements.





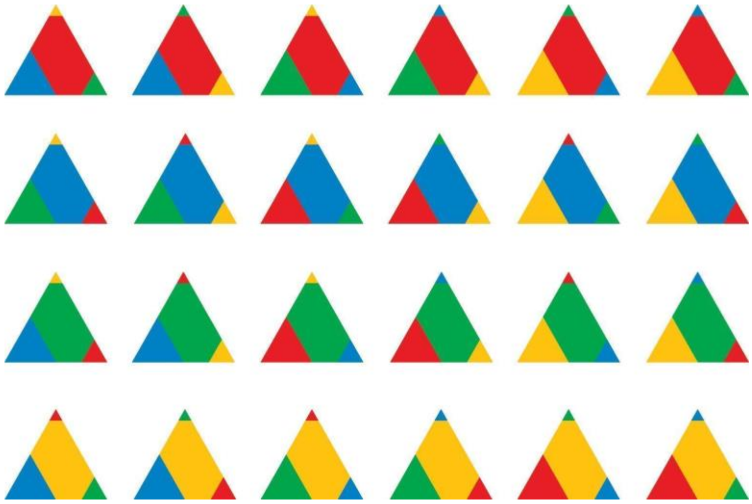
Prior to the Big Bang, the Poly-Universe triangle was condensed into a single galactic hexagon. Not only the basic elements like stars, but also their smaller parts like planets are scattered in outer space.



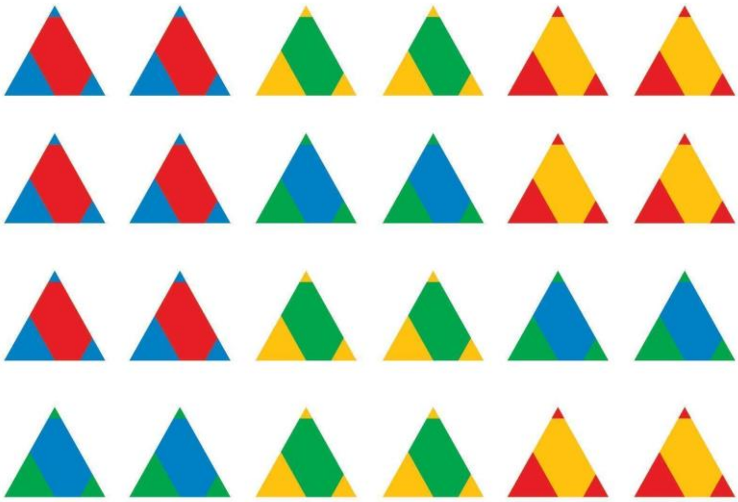
Task: First, find an own planet belonging to all of the stars in the Poly-Universe and then rebuild the galactic hexagon. Can you restore the original state or get a new galaxy? Estimate or figure out quite exactly how many different such there are in the cosmos.

Solution: The first step is to restore the triangular stars and their planets

1 / According to the classic 4 color combinations



2 / The planets are other color than the star body, but they are the same color

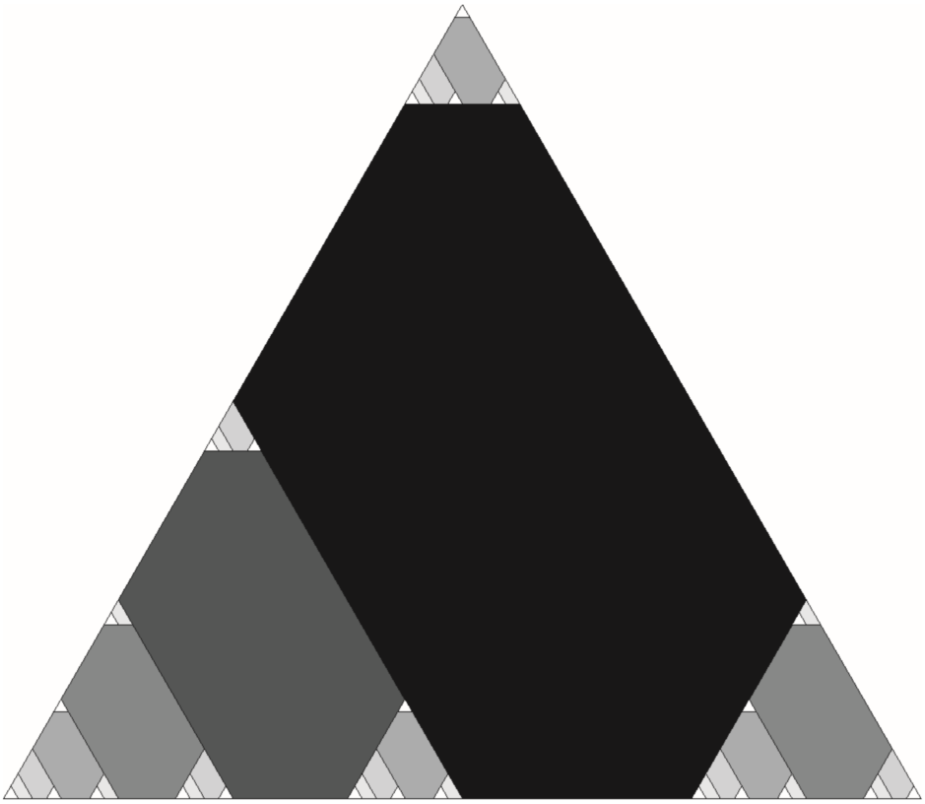


3 / At random

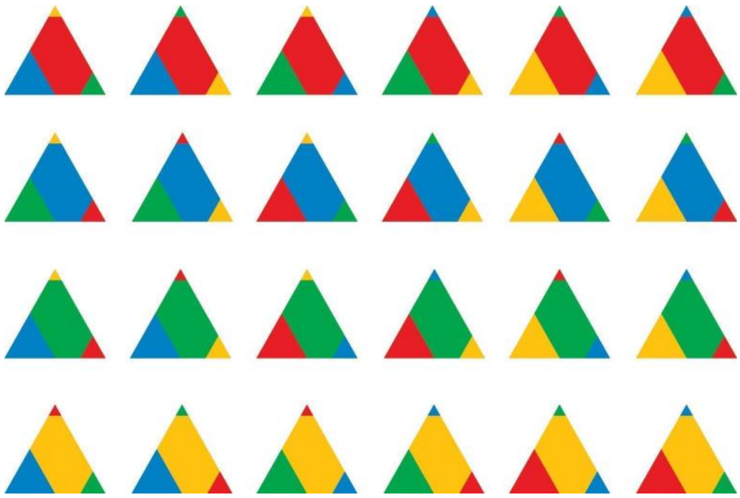
- *Why this exercise is good:* This task developed students' creativity, a complex worldview.
- *Level of teacher training:* Elementary, subject teacher, secondary school
- *School subject(s):* Creative Art, Physics, Cosmology, Informatics
- *Comments:* This task can be outsourced to all ages, but the full solution can only be expected from high school students. Can be solved with colored paper elements or Google Draw or other applications specially designed for this task… It can be done with all three basic forms.

Good practice 22

Poly-Poly Universe Task: In a triangular element similar to the triangular element of the Poly-Universe, Saxon hid the infinitely scaled-down base shape inside the large one.



Thanks to the color combination of the Poly-Universe game family consists of 24 pieces in each basic form, and all of them has same color proportion:

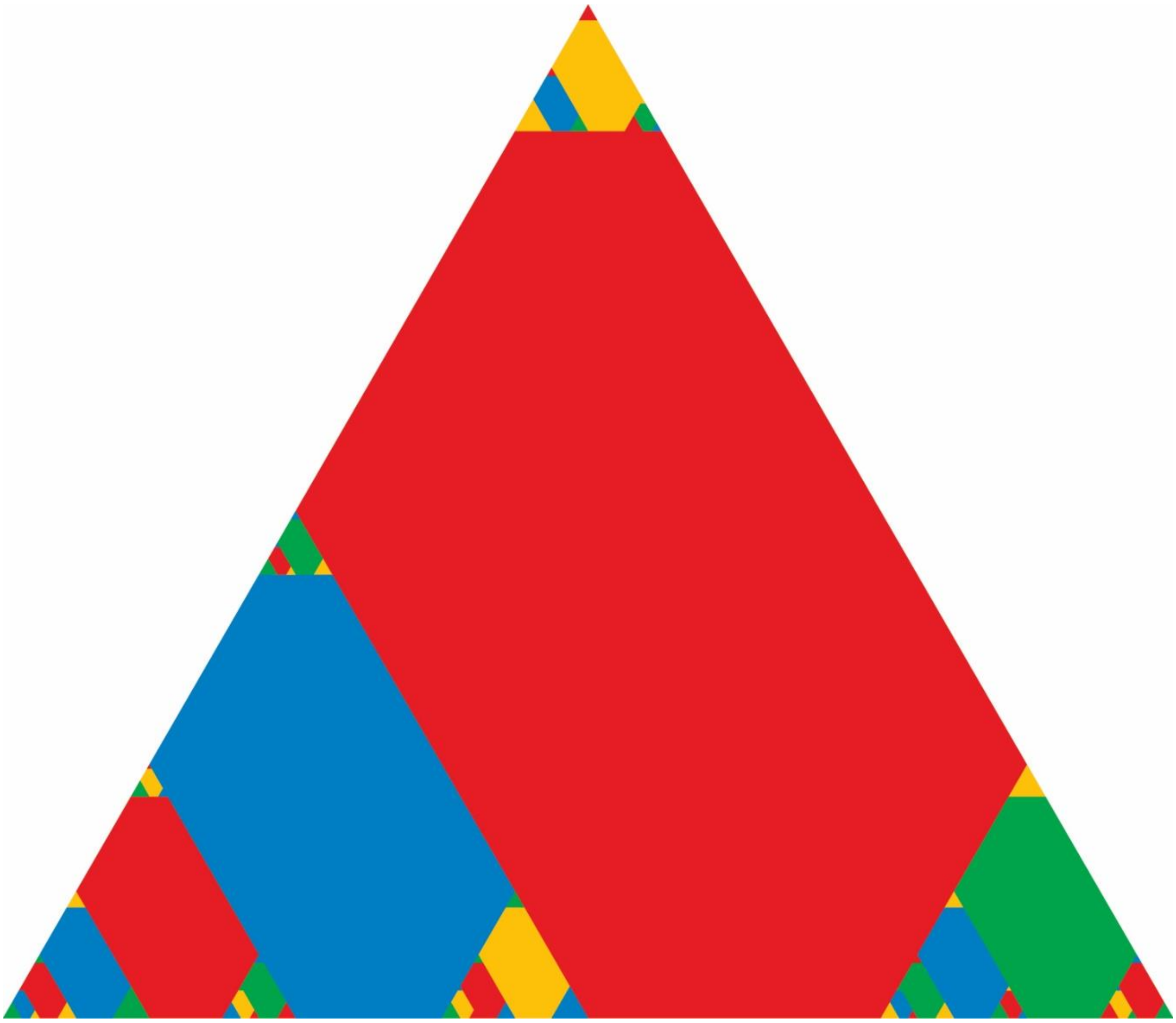


Let's make an attempt to hide all the gradually reduced elements of the element family inside a selected Poly-Universe triangle base element, so that the elements to be hidden are always halved in order to place the smaller elements for full coverage.

The rule according to which the connection is to be made is:

- The base color of the reduced base element must always match the bigger corner shape to which it is being connected.
- The contact color of the corner shapes of the associated base element must not match the color of the receiver forms.

The example below is a possible solution:



Questions that arise:

1. Is there a solution to these rules?
2. How many reductions need to be made so that each element can be placed with the above regularity?
3. How many basic shapes can be placed per reduction?
4. How do the spatial proportions per color evolve in the Poly-Poly Universe thus created?

- *Why this exercise is good:* Develops logical and creative thinking
- L*evel of teacher training:* Subject teacher, secondary school
- *School subject(s):* Creative Art, mathematics, combinatorics, informatics
- *Comments:* We can think of other rules. You need Google Draw or another graphics editor to solve it.

Good practice 23

The relationship between Sierpinski and the Saxon triangle: The Saxon's poly-dimensional artwork below (Figure 1) was created independently of the Sierpinski triangle. In this exercise, we examine the work and find the geometric and mathematical relationships with the classical fractals.

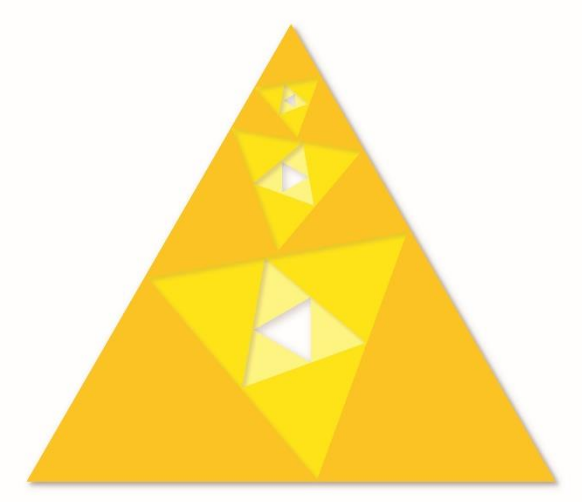


Figure 1: SAXON –Signe 2000, oil on wood 140×130 cm

1. Calculate the visible/remaining area of the two works.
2. Examine how the work differs from the classical fractal.
3. Look for other similar examples in science, art, architecture…

Fig 2: A didactic illustration of SAXON's artwork

Steps in the construction of the Sierpinski triangle.

For the construction of the Sierpinski triangle, an equilateral triangle is usually chosen. However, this is not mandatory; any triangle can be made into a Sierpinski triangle.

This classic algorithm is also used to present the fractal:

1. Take a triangular plate
2. Draw your centerlines
3. Remove the center triangle
4. Repeat these steps for the resulting small triangles

At each step, the side lengths of the resulting small triangles are halved and their area is reduced to a quarter, while the middle triangle disappears.

In fact, the Sierpinski triangle is available as a limit: it consists of the points contained in each iteration step, that is, what remains of the triangle after an infinite number of steps. Computer representations perform the iteration up to ten times because there is no visible change in the next steps for the human eye and the computer screen.

According to the classical area calculation methods, the area remaining in the iteration steps approaches zero.

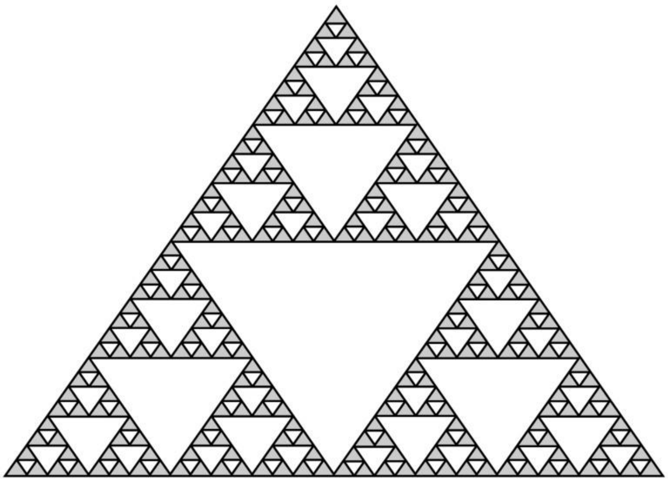


Figure 3: Sierpinski's classic fractal image: *https://en.wikipedia.org/wiki/Sierpi%C5%84ski_triangle*

There is only an aesthetic difference between the Saxon work and the Sierpinski triangle. The artist took advantage of the construction freedom to build the scaling in 6 steps only in the middle bar. He further solved the symmetry effect by tilting the smaller shapes at an angle of 15 degrees relative to each other. In addition, he did not take the smaller-scale forms out of the area but, while retaining his materiality, depicted them in different yellow tones. The Saxon triangle, while carrying the essence of the Sierpinski triangle, in reality reduced the area of the initial figure by only 1/64 + 1/256 + 1/1024.

In the 13th-century cathedral of the small Italian town of Anagni, for example, you can find mosaic tiles similar to the Sierpinski triangle, while there is a 700-year difference between the two ages:

Figure 4: Cathedral of Anagni, Italy, 13th-century

- *Why this exercise is good:* Interdisciplinary approach. The relationship between fine arts, architecture, and mathematics. Relationships between art history and the history of mathematics.
- *Level of teacher training:* Secondary school
- *School subject(s):* Art & mathematics, architect
- *Comments:* Look more than similar examples in science, art, architecture…

Good practice 24

After a theoretical explanation the student can analyze the masterpieces of cubism in a museum, or their reproductions.

This art movement can be presented to students in such a way that students can paint a picture and after that they can make the cubistic version of that painting using Poly-Universe sets. As they can remove and change the tiles, they can choose the version, which they like best.

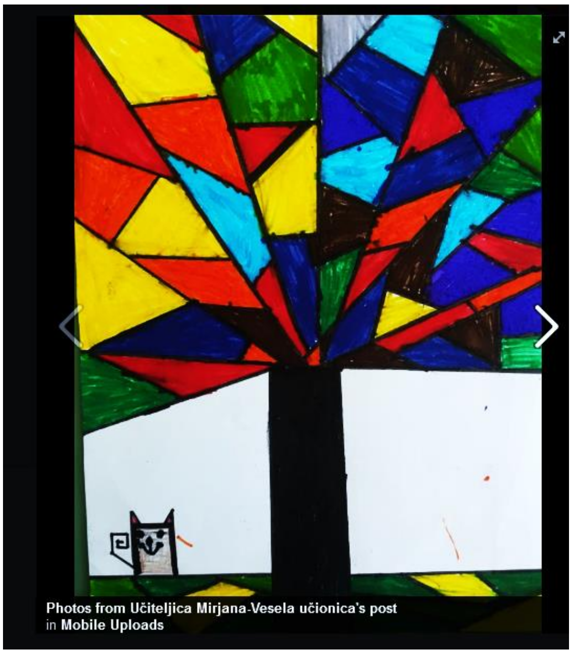


Figure:
https://www.facebook.com/uciteljicamirjana/photos/pcb.2362205114040742/2362204817374105/?type=3&theater

An illustration of a cubist picture is presented without the painting from which it originates. Our students will make similar pictures by using Poly-Universe sets.

Finally, they can calculate the sums of the surfaces of tiles painted in the same color and can determine the percentage of the area of tiles painted in the same color as the area of all the tiles used.

Variations:

a) A picture can be presented to all the students and each of them can create a cubistic version of it by using tiles. The winner is the student whose picture has the largest (the smallest) area of tiles.

b) The students put together a cubist version of the given picture.

- *Why this exercise is good:* Competences which are developed and knowledge which is deepening: creativity, aesthetic competence, organizational skills, communication skills, collaboration skills.
- *Level of teacher training:* Elementary, secondary school, subject teacher
- *School subject(s):* Art and design, languages, mathematics, history
- *Comments:* For ages over 10 years

Good practice 25

Designing geometric patterns for a carpet inspired by the Pirot rug, the manufacturing of which is included into the list of Intangible cultural heritage of Serbia

The Pirot rug is a hand weaved woolen carpet with unique motifs and ornaments having both front and back sides identical.

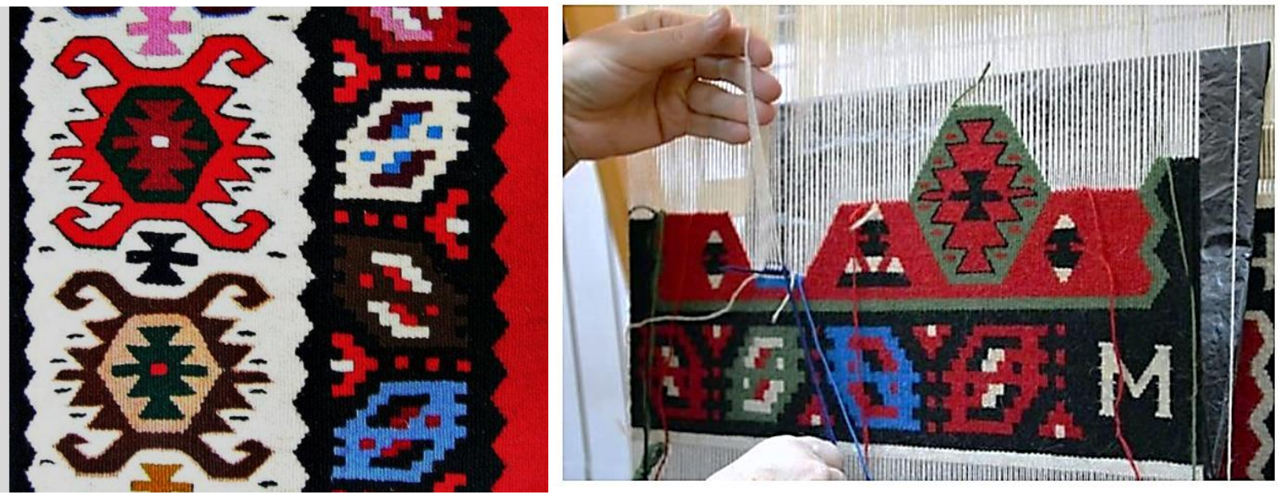


Figure: *https://srpskatelevizija.com/2017/05/24/cilim*

All the ornaments have their special meanings and it can be the starting point for discussion, exploration and analyzing of folk customs, beliefs, tales and (or) poems.

Using Poly-Universe sets and inspired by ornaments of Pirot rug the student can design new geometric patterns for their carpet of 30 cm long and 20 cm wide.

If we know that ½ g of wool has to be used for weaving one square centimeter of Pirot rug, please, calculate the mass of wool in each color, which is necessary for making the carpet.

The students learn weaving and they can weave their carpet.

The geometric patterns of the Pirot rug can be compared with the geometric shapes of traditional handicrafts of other nations.

- *Why this exercise is good:* Competences which are developed and knowledge which is deepening: creativity, aesthetic competence, organizational skills, communication skills, collaboration skills, weaving
- *Level of teacher training:* elementary, secondary school, subject teacher
- *School subject(s):* Art and design, languages, mathematics, history
- *Comments:* For ages over 10 years

Good practice 26

Rhythm written in tiles. Doing rhythmic exercises students can use Poly-Universe tiles in the following way: When they sing a song written using *musical* notes they can additionally use the same instruments to emphasize the rhythm of the song.

For example, a song: Twinkle, twinkle little star which text was written by Jane Tajlor:

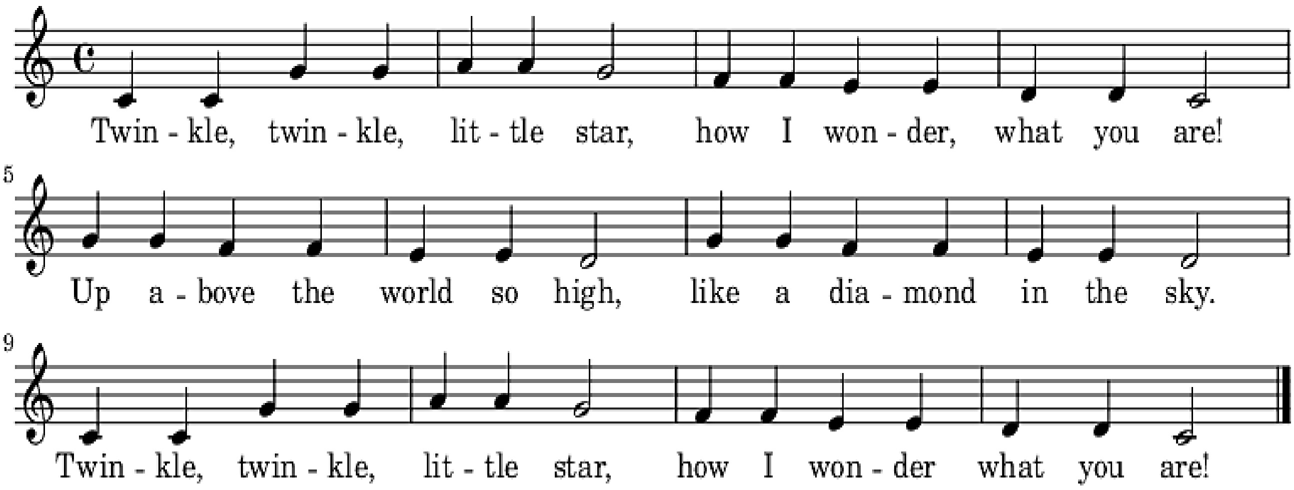


Figure: *https://upload.wikimedia.org/wikipedia/commons/1/16/Twinkle_Twinkle_Little_Star.png*

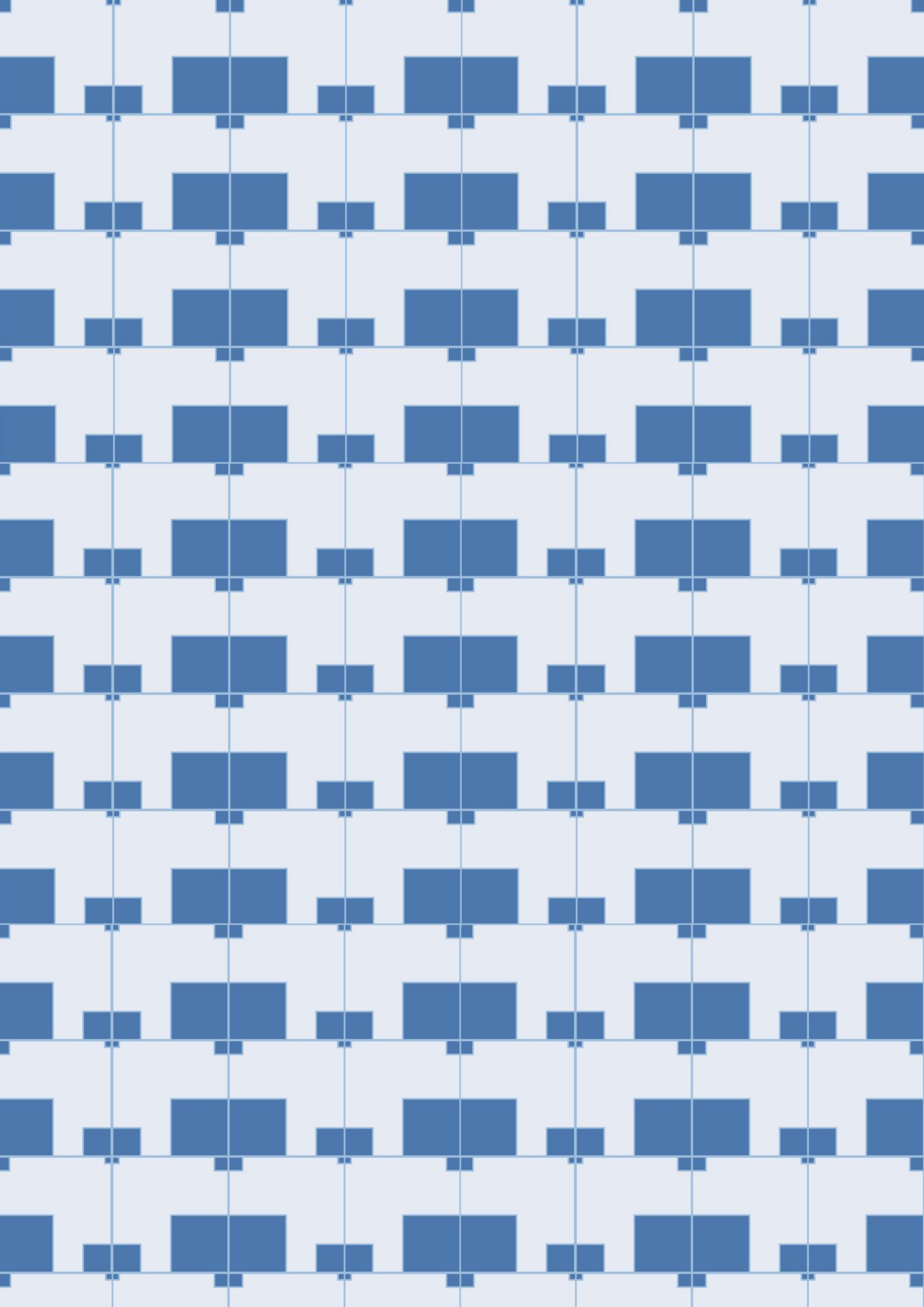
Rhythm can be given by different instruments: cymbal, triangle, rattle, and clapping. For each line of the song notation, clapping is used for notes of the same (quarter) duration, and for longer note durations (halves), rhythmic instruments are used: triangle in the first row, rattle in the second row and cymbals in the third row. In order to indicate which (notation the using) of these instruments is applied, the following tiles can be used: triangle can be represented with a tile which is triangular, the rattle can be represent with a half of a circle, cymbal can be represented using a square and the clapping can be represented with dodecahedron (from the 'square' set). The musical notes can be supplemented with the notation of rhythm by placing the adequate tile under each note and the corresponding text. (A photo of the students' work is needed.)

The students can make their own versions using rhythmical instruments.

After analyzing the text of the song the students can use the tiles to create a picture which can represent the meaning of the text.

The students can use the patterns by writing down the rhythm for creating their carpets as well.

- *Why this exercise is good:* Competences which are developed and knowledge which is deepening: creativity, aesthetic competence, communication skills, collaboration skills, sense of rhythm.
- *Level of teacher training:* Elementary, secondary school, subject teacher
- *School subject(s):* Music, art and design, languages, mathematics
- *Comments:* For ages over 10 years



5.3 Good practices — Interdisciplinary Approaches

Good practice 27

Task could be realized within the following steps:

Step 1: Students gain basic knowledge about the morphology of flowers of different plant families.

Step 2: Students analyze the photographs, schemes or fresh plant materials and increase their knowledge about the morphology of flowers of different plant families.

Step 3: By using the Poly-Universe sets students create the models of flowers of different plant families.

Step 4: Students present their models, discuss their solutions and provide feedback information to each other.

Some examples of possible solutions which students can create as a model of different plant families are presented in Figure 1.

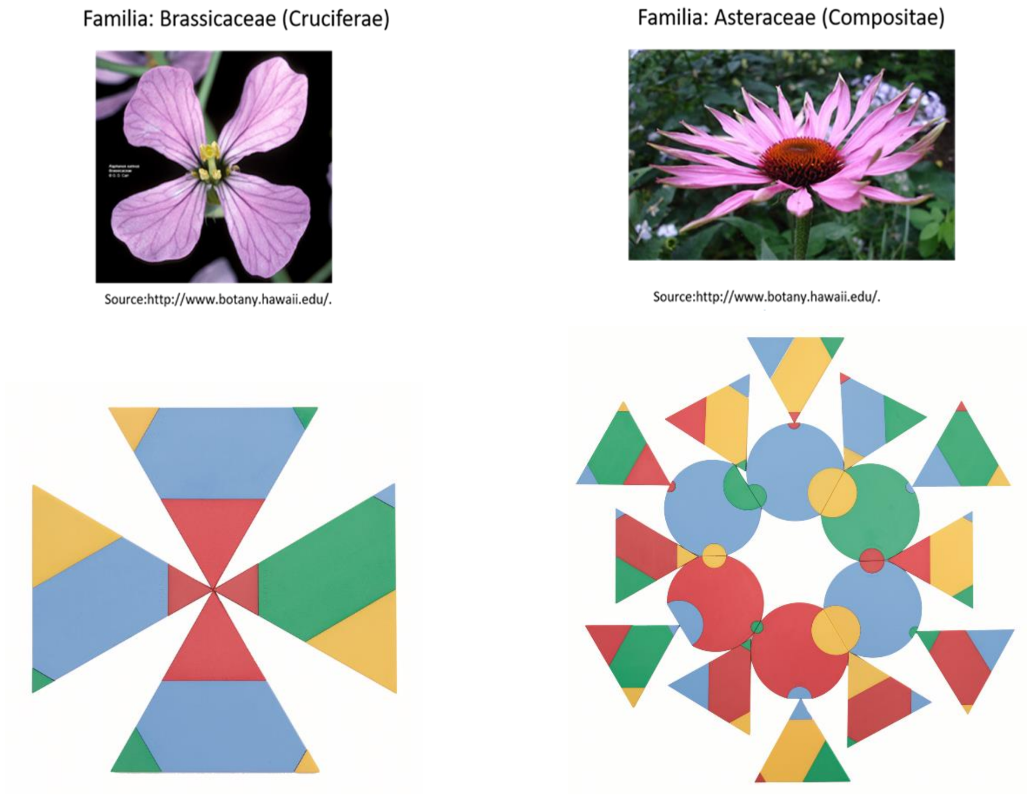


Figure 1: Possible solutions for plant families model creations from Poly-Universe sets.

- *Why this exercise is good:* This activity encourages student's creativity, knowledge implementation, fast feedback by peers, interdisciplinary learning and communication between the students.
- *Level of teacher training:* These activities could be used for biological- botanical education in primary and secondary school. Basically, simple and most common plant families could be a task for primary school students and more complicated and families could be a task for students in secondary school.
- *School subject(s):* Biology, math, art

- *Comments:* This task can be easily adapted to the students with disabilities. For example, teachers can provide photos of created models of plant families and assign them tasks to create the model by using the Poly-Universe sets. On the other hand, teachers can increase the cognitive level of the task for gifted students. Teachers can set rules for gifted students, such as connecting only the same shape or color in the model creation. In this way, the task can be more challenging for students and allow them to fully develop their potential.

Good practice 28

Description: Enzymes are molecules, most often proteins, which with their biocatalytic action help to convert a certain substrate (analyte) into a product. The substrate is the molecule on which the enzyme acts, while the product is the product of a catalytic reaction. Enzymes accelerate chemical reactions in which they do not participate themselves, reducing the activation energy. The substrate on which the enzyme will act must be a specific type of molecule, otherwise there is no catalytic reaction. Enzyme selectivity is based on a key-to-lock model. The lock is represented by an enzyme, while the key is a substrate.

Task: Students are divided into two groups. The first group using Poly-Universe creates a substrate, based on their role the second group creates an enzyme. The enzyme and the substrate must fit or there will be no product needed to be created in the body. The time for which the other group produces the enzyme from Poly-Universe is also measured. After this the groups change roles. The second group makes an enzyme based on the substrate that the first group will make. Some examples of possible solutions which students can create as a model of substrate and enzyme are presented in Figure 1.

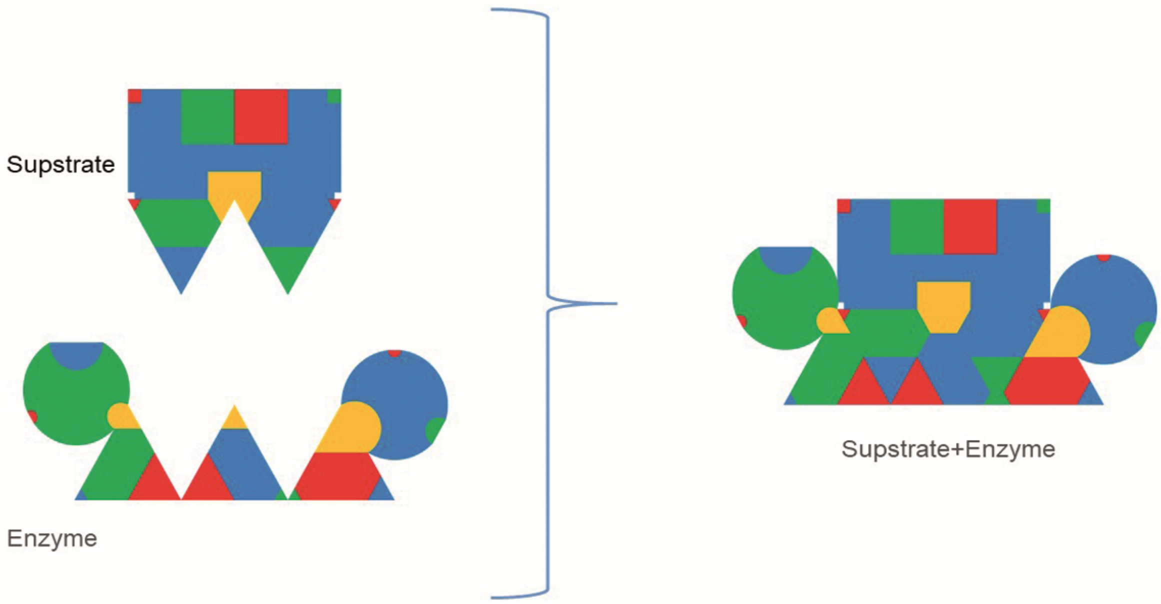


Figure 1: Examples of substrate and enzyme connection created from Poly-Universe.

- *Why this exercise is good:* This activity encourages student's creativity, knowledge implementation, increase the presence of gaming in the learning process, interdisciplinary in learning and communication between the students
- *Level of teacher training:* Secondary school, age 12–14.

- *School subject(s):* Biology, math, art
- *Comments:* The teacher can adapt this activity to the students with disabilities if they allow them at the beginning to create the substrate, and without any additional rules. On the basis of the substrate which is created by students with disabilities, other peers can create the enzyme. When students with disabilities successfully achieve this task, the teacher can assign them a task to create an enzyme at the basis of the very simple substrate. If teachers want to adapt this task for gifted students they can set rules in the creation of substrate and enzymes should be connected parts with the same colors, size and shape.

Good practice 29

At the end of the learning process, students will be able to understand the life cycle of a butterfly, describe each step in the life cycle and know the differences and similarities between different steps in the life cycle.

The task could be realized within the following steps:

1. Students in groups or pairs analyze the video or text in which is described the life cycle of a butterfly.
2. Using the Poly-Universe and plastic straws students create the life cycle of butterflies.
3. Students present their models and explain them to classmates.
4. Students discuss different solutions and provide feedback to each other.

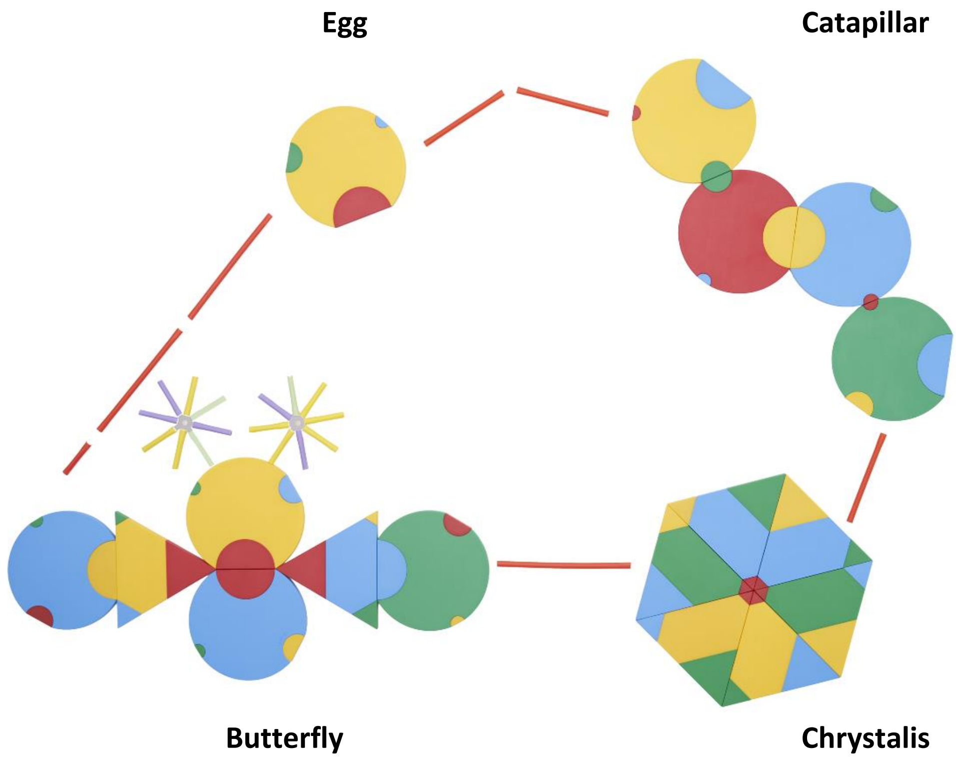


Figure: One possible solution is presented in the photo below.

- *Why this exercise is good:* This learning task helps the students to understand the process of the life cycle of butterflies within the practical implementation of their knowledge and develop their creativity.

- *Level of teacher training:* Primary school
- *School subject(s):* Biology, math, art
- *Comments:* Teachers can adapt this task to the student with disabilities by giving them a photo of an already created model of a butterfly from Poly-Universe and with direct instructions assign them a task to rebuild it. If teachers want to increase the level of difficulties for the gifted students, they can set their rules for connecting parts of Poly-Universe, or ask a student to set their own rules for connections during the process of creation.

Good practice 30

In biology, symmetry is the orderly repetition of parts in a living being. Simply looking at an organism reveals its external symmetry. There are several types of the symmetry of animals such as: radial symmetry, bilateral symmetry, spherical symmetry...

At the beginning of the activities, the teacher should provide an explanation about the animal symmetry to the teachers. This information could be in the form of a text, film or teacher's presentation. In this task, students should create different types of symmetry in the animal kingdom by using Poly-Universe.

Students should present their works and share their ideas with classmates. Through the discussion students should provide feedback to each other and if that is necessary provide corrections to their peers.

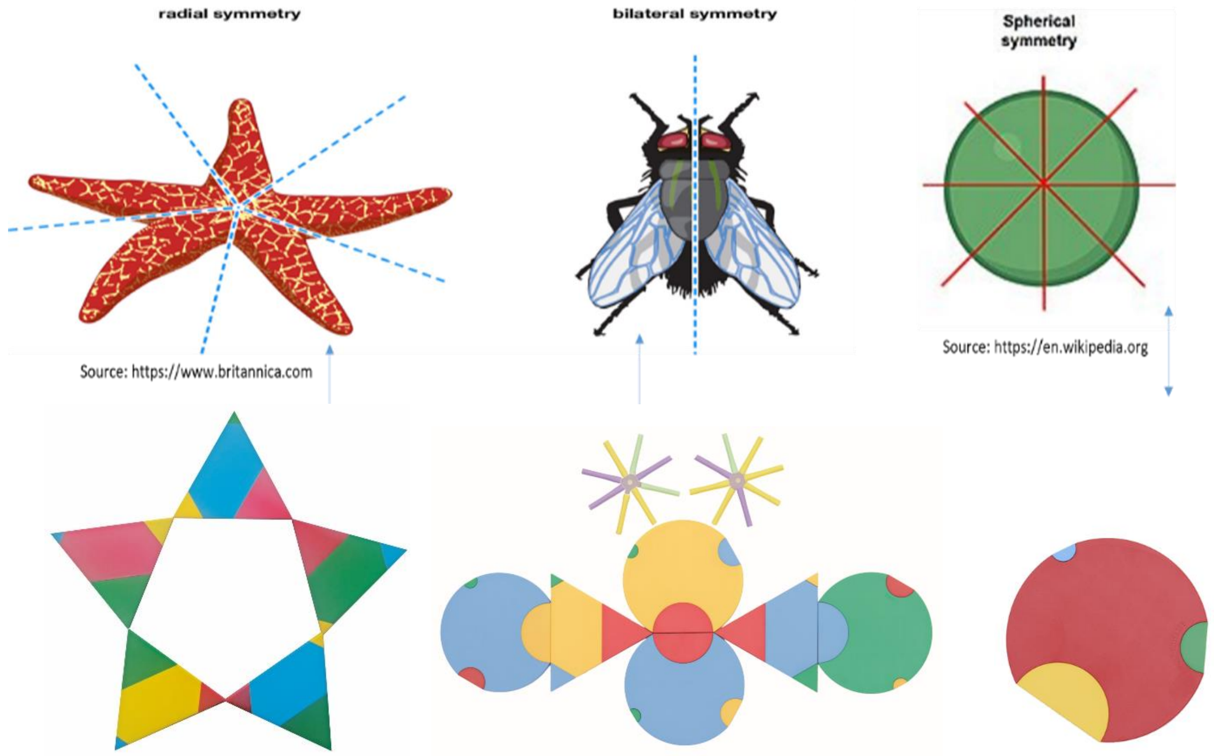


Figure: Some possible solutions are presented in below

- *Why this exercise is good:* Using the geometrical shape in the Poly-Universe the students can during the hands-on activities gain the knowledge about animal symmetry.

- *Level of teacher training:* Elementary school
- *School subject(s):* Biology, math, art
- *Comments:* The teacher can adapt this task to the student with different abilities by assigning them to create one or more animal symmetry from Poly-Universe. Less able students can create a small number of symmetry, contrary to gifted ones who can create more examples of animal symmetry.

Good practice 31

Task could be realized within the following steps:

Teachers provide information for students about molecules they learn about, for example oxygen molecules, water molecules, sulphur dioxide, nitrate and so on. This information could be in the form of teachers' presentations, text, films or a combination of all of this.

Students analyze information and gain knowledge about these molecules. After that, using Poly-Universe and plastic straws, students create the model of molecules about which they learned.

Afterwards, students present their models, explain them, discuss the models and provide useful feedback to each other. Examples of models which could be created are presented.

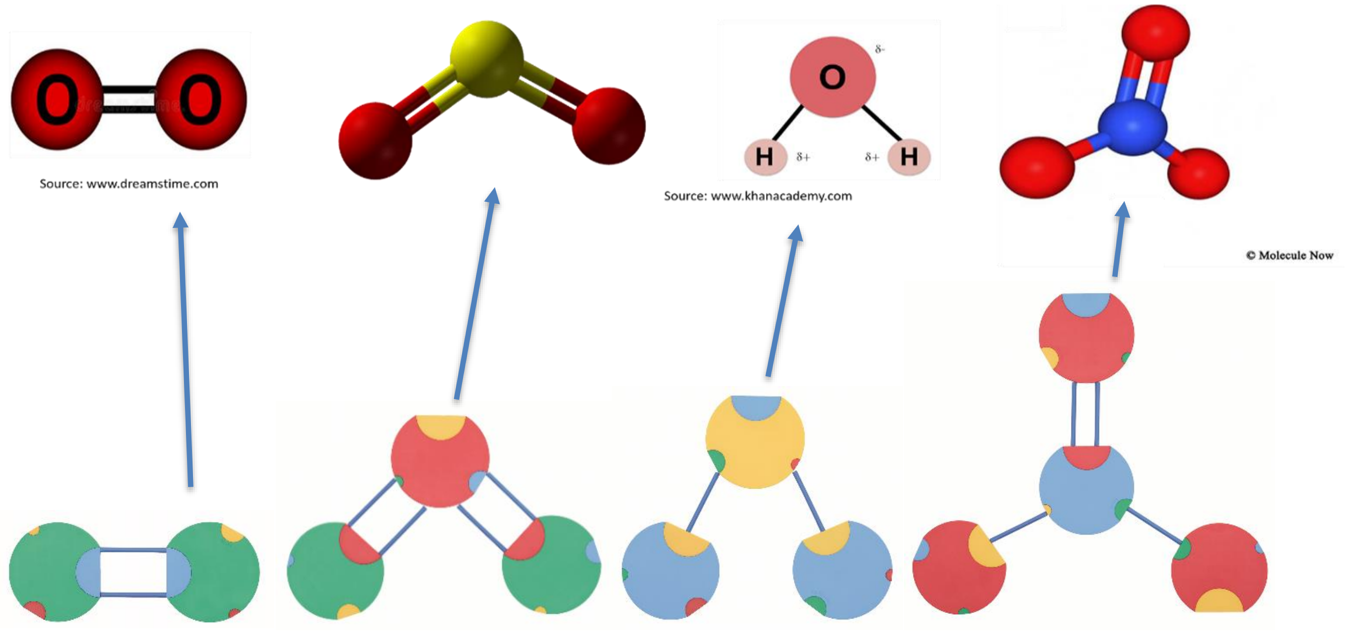


Figure: Examples of molecules model created from Poly-Universe.

- *Why this exercise is good:* This activity encourages students' creativity, knowledge implementation, increases the presence of gaming in the learning process, and is interdisciplinary in learning and communication between the students.
- *Level of teacher training:* These activities could be used for chemical education in primary and secondary school.
- *School subject(s):* Chemistry, math, art
- *Comments:* This activity easily could be adapted to the students with disabilities. For example, teachers can prepare a 3D model of the molecule for these students, and ask them to create the same from the Poly universe. For gifted students, teachers can adapt the task by setting them the rules for Poly-Universe part connections.

Good practice 32

The teacher provides students with information about the connections between the bones. The bones are connected by joints, on the structure of which the mobility of different parts of the body depends. The fixed connection between the bones is called the suture. In this way, the bones of the skull are connected. The serrated ends of one bone fit into the serrated ends of the other and thus grow together, closing the skull that protects the brain). Some bones are connected by cartilage – it is an elastic, poorly mobile connection. In this way, the ribs and sternum are connected, as well as the spinal vertebrae. In the movable joint, usually, one bone has a bulge – the apple, which enters the recess – the cup, the other bones.

Students analyze this information as well a picture on which bone connections are presented and with using the Poly-Universe create the models of connections.

After the creation, students present their models to the rest of the classmates, which discuss them and provide feedback. Simple models created from Poly Universe are depicted in the photo below.

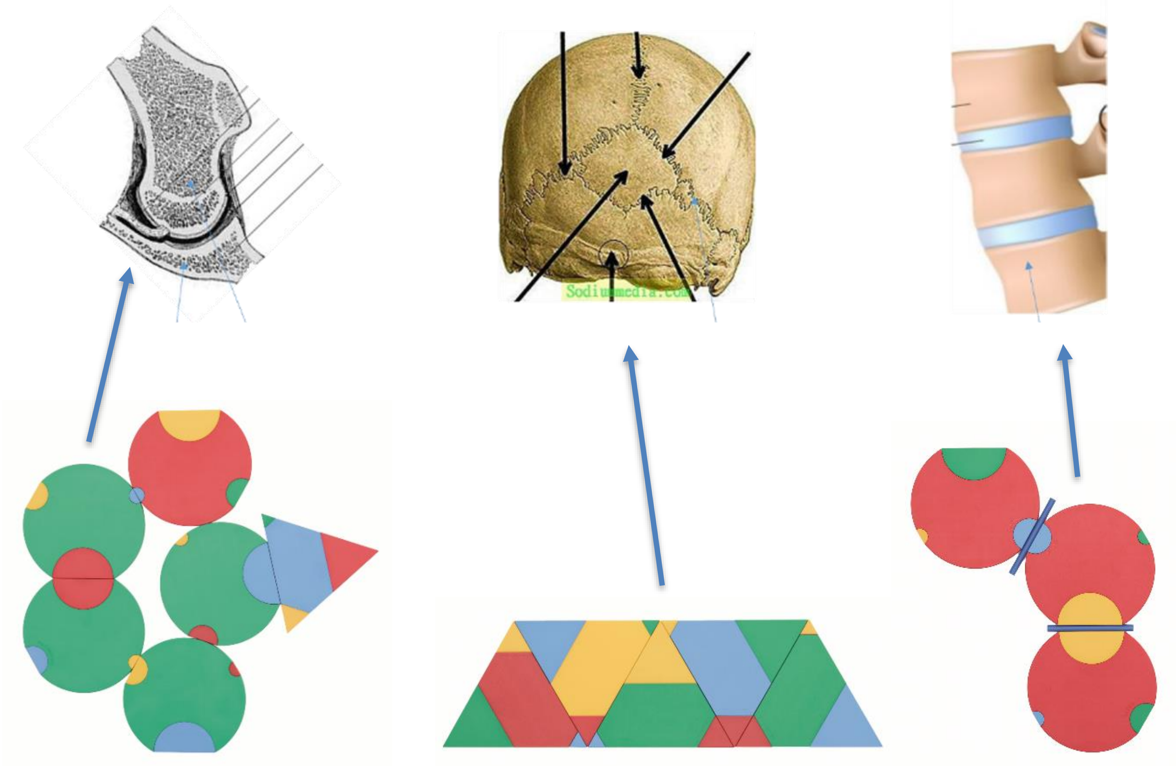


Figure: Some possible solutions are presented in below

- *Why this exercise is good:* This exercise contributes to the students better understanding the connection of the bones in the human body, allows the students to express their creativity and implement knowledge from the different subjects during the learning.
- *Level of teacher training:* Elementary school
- *School subject(s):* Biology, math, art
- *Comments:* Teachers can adapt this task to the students with disabilities on the way to provide to them photos, 3D models and models from Poly-Universe on which are presented types of connections between the bones and assign them to connect the models and photos which present the same type of bone connections. To the gifted students, teachers can assign additional tasks, such as trying to find a place in the body where the bone connection which you are created is presented.

Good practice 33

Task could be realized within the following steps:

Teachers provide to the students the following information: Skeletal muscle can be seen as a bundle of grouped tissue, connected into a complete whole. The smallest part of this tissue – the base of the working muscle – are the filaments actin and myosin. The filaments are grouped into myofibrils. Myofibrils are further grouped into muscle fibres, smaller muscle fibres are grouped into larger ones. Muscle fibres group themselves into bundles wrapped in connective tissue. A smaller muscle can consist of only a few bundles of fibres, while a larger muscle can consist of hundreds of bundles of fibres. The teacher should also provide to the students a schema that presents a skeletal muscle structure. Teachers can present this information in the form of printed materials which students should analyze or in the form of teacher presentations.

- By using the Poly-Universe students create the structure of the skeletal muscle model.
- Students present their models and explain them, with a clear indication of units in the skeletal muscle structure.
- Students discuss the models and provide feedback to each other.

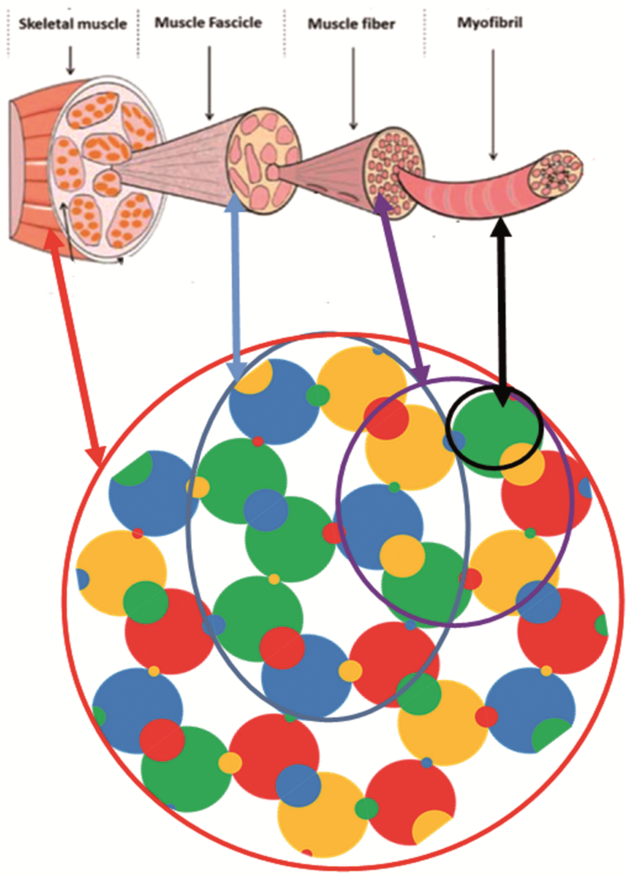


Figure: One possible solution for this task

- *Why this exercise is good:* During the working on this activity, students will gain the knowledge which is required to understand the structure of skeletal muscle. This activity allows the students to express their creativity and implement previous knowledge and experience during the learning process.
- *Level of teacher training:* Secondary school
- *School subject(s):* Biology, math, art

- *Comments:* Teachers can provide an additional task for gifted students, asking them to find fractals in the skeletal muscle structure as well as in their model. If necessary, teachers can adapt this task to students with disabilities by making a model for them and instructing them directly to make it by observing the model of the teacher or one of the classmates.

Good practice 34

At the end of the learning, students will be able to distinguish the types of compound leaves and recognize them on plants.

Task could be realized within the following steps:

1. Teachers provide information about compound leaves morphology and examples from herborized or fresh plant materials to the students.
2. Using the Poly universe and plastic straws students create some types of compound leaves
3. Students present their models to their classmates and explain them.
4. Students discuss the presented model, and provide feedback to each other.

One example of possible compound leaf creations made from Poly universe is given in the photo below.

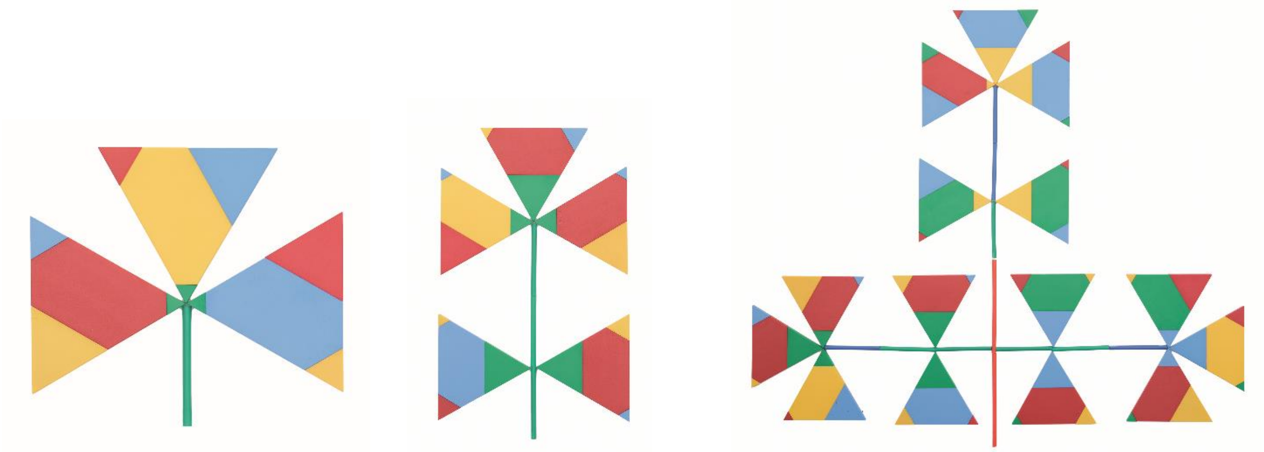
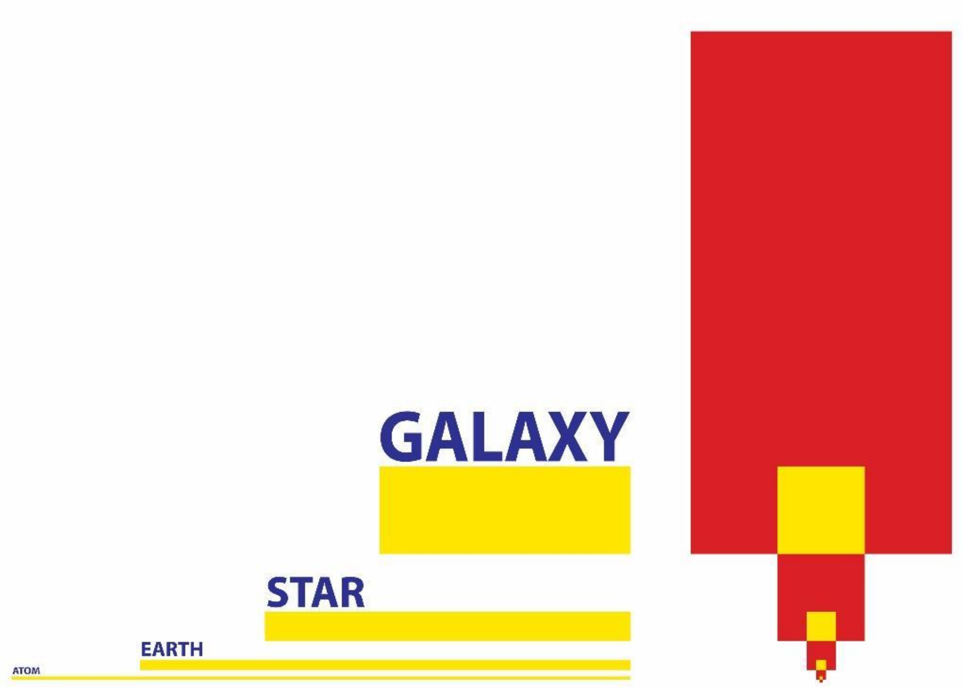


Figure: Examples of compound leaves created from Poly-Universe

- *Why this exercise is good:* This exercise allows the students to interact with different learning materials such as textbooks, plant material, Poly-Universe. Practical activities in the exercise develop student’s creativity and connect it with students’ surroundings.
- *Level of teacher training:* Elementary school
- *School subject(s):* Biology, math, art
- *Comments:* Teachers can organize this activity in the schoolyard, and students can create the model of a compound leaves of plants in their surroundings.

Good practice 35

The Dimension Pencil as an imaginary tool in the hand of an artist: According to Saxon with our dimension pencil, we can to draw a line between the unshakeable Galaxies, on the surface of our roamable Earth and around the buzzing Atoms in same time. This fantastic possibility may incite us to find our real place in the real universe. The question is how is it possible?



Let us travel into the world of elemental (atomic and subatomic) particles, explore all the possible ranges known so far, align them so that we take into account not only differences in size but in magnitude. How many such ranges exist, and at what depth and occasionally how thick should the tip of our Dimensional Pencil be in order to leave a mark on the ‘surface’ of each elementary particle?

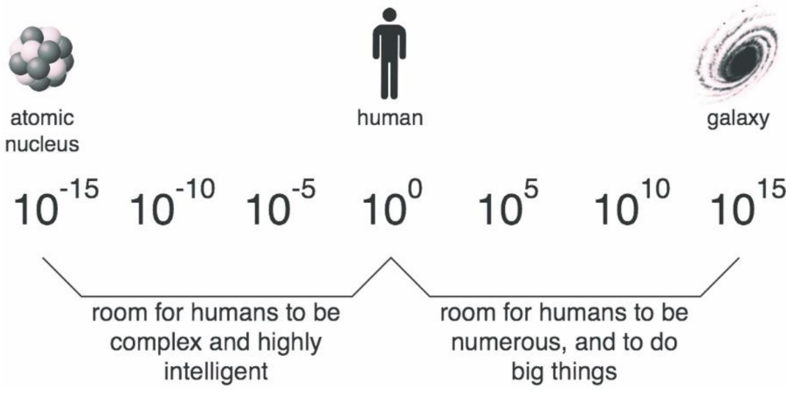


Figure: *https://www.gabrian.com/the-scale-of-universe/*

- *Why this exercise is good:* A series of questions draws attention to differences in the magnitude of the microcosm. Once answered, attention can also be directed to the macro worlds for the coverage of the entire universe, in the process of examining the orders of magnitude relative to each other.
- *Level of teacher training:* Subject teacher, secondary school
- *School subject (s):* Chemistry, mathematics, atomic physics, cosmology

- *Comments:* After the atomic world, the next step of the scale is not our material world on Earth, but our solar system can be: https://hvg.hu/instant_tudomany/20151009_vilagegyetem_meretei_video

Good practice 36

Relationship between human hair thickness and length and poly-dimensional works created by dimensional compression.

One of the special creative methods of the inventor of the Poly-Universe Game is ‘dimension compression’.

In the first step it consists of a basic form, see ‘Dimension Antennas’ artworks below. The artist continuously reduces one dimension / extent of the plain, while in the work leaving it free to run in the other direction in order to retain its area. The square, meanwhile, becomes a rectangle, then a thinner and thinner line-like shape.

Task 1: Our starting square is 10×10 cm

In how many steps can we achieve the thickness of human hair at 1/2 and 1/3 pressing? How long will our imaginary hairline then be?

Task 2: Measure the length of your neighbor’s hair in centimeters, measure it in micrometers, or estimate its thickness based on average data.

Let’s reverse the first line of thought and imaginatively condense all of her/his hair, back into the plane. Calculate and draw how big squares can be created if we have used all the hairs of our fellow partner in our imagination, and covered the shape with it.

Task 3: Make your own art creation in a small group of 3–4 people with this creative method from paper and yarn, until it just fits into the classroom, then next step needs exit to the corridor of school.

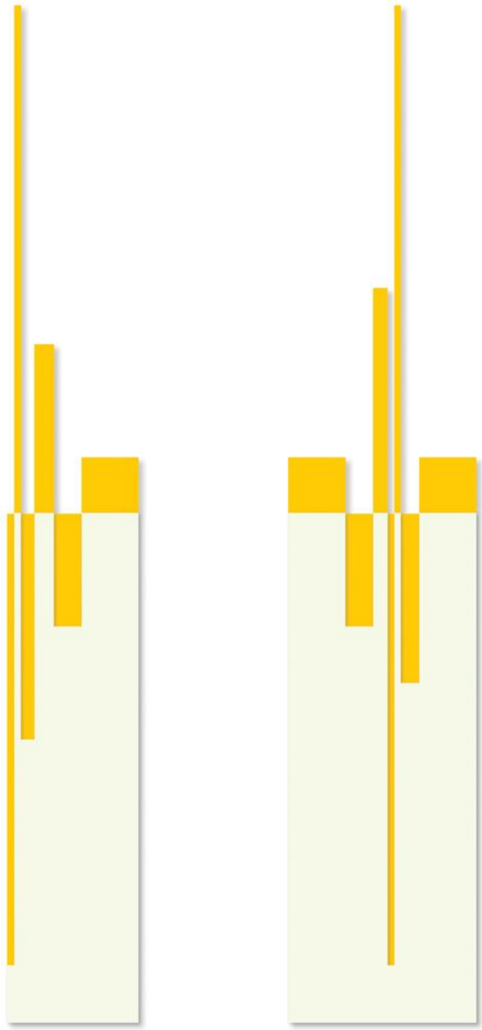


Figure : SAXON – Dimension Antennas I-II, 1999, oil on wood 40x200 cm

- *Why this exercise is good:* Relationship between human hair thickness and length and poly-dimensional works created by dimensional compression. Anthropomorphisation.
- *Level of teacher training:* Subject teacher, secondary school
- *School subject(s):* Biology, anthropology, mathematics, technology, creative arts
- *Comments:* To solve this question, students should look at the data while taking the measurements: Blonde hair thickness = 0.05 mm and an average of 150,000 hair. Dark hair thickness = 0.2 mm (4 times the previous one) and the wearer has an average of 110,000 hairs. Hair colors other than this are worth estimating between the two extreme data.

Good practice 37

Figure 1: SAXON – Space Glider 2008, 50x50x700, metal public artworks

The picture above shows a public space work of Saxon, the title is 'SPACE GLIDER', which is based on the principle of dimensional compression. Why do you think the creator gave him this title? Saxon also made a similar work with the proportions of the square element of the Poly-Universe, which he transplanted from the plane into space.

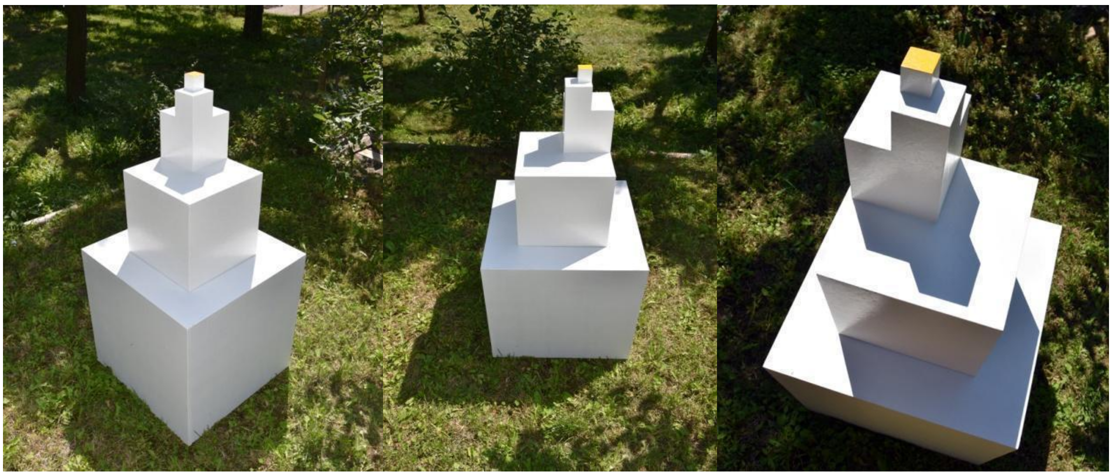


Figure 2: SAXON – Dimension Tower, 50x50x70 cm, oil on compressed wood

Now we want to design a poly-dimensional tower (Figure 2-3) and build an elevator in it that will take us directly to the moon. The floor area of our building is square-based, from the corner of which the elevator starts. The building is designed so that we always start another square column from the next level, the base area of which is a quarter and the height of which is the volume of the original cube. We repeat this process until our tower house reaches the surface of the moon. Model it with wood and cardboard box in a few steps until it fits in the classroom…

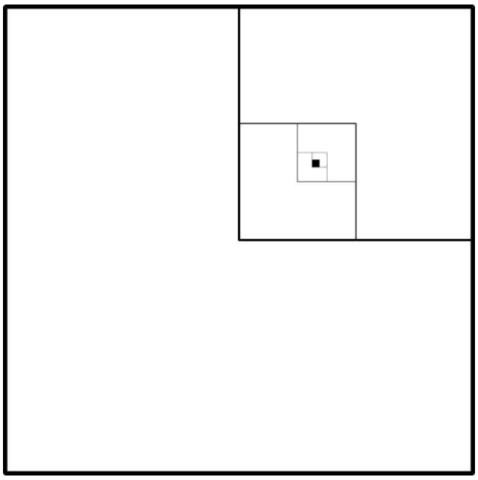


Figure 3

Questions:

1. Which are the tallest buildings in the world today, and after how many pressing step does the height of the poly-dimensional tower leave them.
2. How many steps can we get to the moon? If you need to change per level, how many transfers will be needed? Did you know, in Eiffel Tower also needs to be transferred level by level to the next higher lift?
3. How fast does the elevator have to be to get there in an hour?
4. After how many levels do we reach weightlessness?
5. What is the floor area of the starting building/square, in order to reach the Moon for get off? Think about being able to fit in the elevator....

- *Why this exercise is good:* Complex questions, really interdisciplinary task
- *Level of teacher training:* Subject teacher, secondary school
- *School subject (s):* Architecture, design, astronomy, mathematics, physics, technology
- *Comments:* The question set can also be asked in larger perspectives

Good practice 38

Making a pattern for memorizing the right order of the words in a sentence in a foreign language (for example: English). Each type of the words can be represented by a tile. For example: subject, object and the verb in a sentence can be represented by 3 big tiles of different shapes. Each type of adverb can be represented in different colors by small tiles in the same shape as the representation of the verb. Such representations emphasize that adverbs refer to the verbs. The representation of the whole sentence is a string of tiles and it can help students to memorize the right order of the words. Let us look at the following sentence:

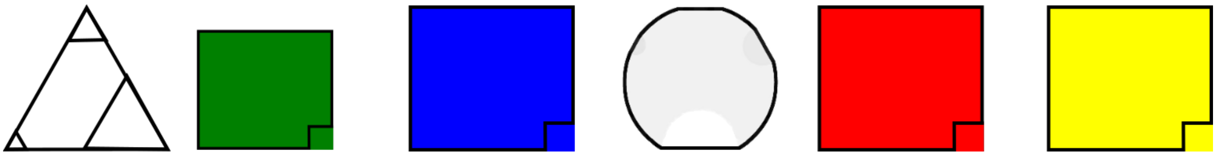
Students often play music in the club in summer.

The sentence can be analyzed and represented this way:

	Subject (who/what)	Adverb of frequency (how often)	Verb (action)	Object (complements)	Place (where)	Time (when)

Present Simple	Students	Often	play	music	in the club	in summer

The right order of the words in an English sentence can be represented in the following way (such a way):



Students can change the positions of the tiles or they can remove them from the string of tiles. Then they can discuss the meaning of the new sentences (They can also conclude that the new sentence is meaningless).

Students can calculate the number of sentences (meaningful or meaningless) formed using subject and verb and some of the adverbs of the sentence mentioned above.

- *Why this exercise is good:* Competences which are developed and knowledge which is deepening:
- creativity, communication skills, collaboration skills, foreign language grammar
- *Level of teacher training:* Elementary, secondary school, subject teacher
- *School subject(s):* Foreign languages, mathematics
- *Comments:* For ages over 10 years

Good practice 39

The Poly-Universe tiles in chemistry. The Poly-Universe tiles can be used for representing chemical reactions. For example:

$$NaOH + HCl \rightarrow NaCl + H_2O$$

Atoms and their representations	Na	O	H	Cl

The students represent the molecules of the substances that react with each other using tiles:

$NaOH$	HCl

Then they regroup the tiles and in this way they represent the chemical reaction. As a result they get the new substances:

$NaCl$	H_2O
	

Analyzing the representations above the student can write down the chemical reaction using symbols.

- *Why this exercise is good:* Competences which are developed and knowledge which is deepening: creativity, communication skills, collaboration skills
- *Level of teacher Training:* primary school upper grade, secondary school
- *School subject(s):* Chemistry
- *Comments:* For ages over 12 years

Good practice 40

Finding the constructed building. The game is played in two small number groups. Each group has its own table of 9 squares (3 in each row and column; sides of the squares are 9 cm long) plus two different Poly-Universe sets. The students of groups create their buildings consisting of towers which are made by placing the next tile over the previous one and all the tails of a tower have the same shape and size. The height of the towers is maximum 4 tiles and a tower can be built on a square of the table. The buildings cannot be seen by the members of the other group (the members of groups can sit in two lines turning their backs to one another.)

The members of the groups alternately pose questions to members of the other group about the building of the other group. The answer can be only: 'yes' or 'no'. '

The students of one group have to share their duties among the members of the group and have to collaborate in order to find out what the building of the rival team looks like. They also have to decide on the way of noting down the results of the answers which they get from the members of the other group.

- *Why this exercise is good:* Competences which are developed and knowledge which is deepening: creativity, problem solving, communication skills, collaboration skills and organizational skills.
- *Level of teacher training:* All levels
- *School subject(s):* All subjects
- *Comments:* For ages over 7 years

Good practice 41

'Telling a story' game. The game is played in four number groups with one or two sets of square shape tiles. The concave dodecagons (colored) in four different colors can represent four different verbs; big squares can represent four different nouns; middle squares different adjectives and small squares different conjunctions. One of the players deals all the tiles to the groups. The aim of the game is to create a funny story by clubbing together in such a way, that someone from the first group places a tile on the table and says a word of the adequate type. The second group has to create a sentence using this word. A member of the group has to read the sentence and put another tile on the table and say the proper word. The next group puts this word into the second sentence of the story and so on. If a group gets a tile which is identical with one of the tiles on the table, then they will use the same word that was used previously. Because of that the members of the group have to make notes of the key words. The game is over when all groups play their last tile. At the end of the game the players can analyze and discuss the created story.

- *Why this exercise is good*: Competences which are developed and knowledge which is deepening: creativity, problem solving, communication skills, collaboration skills and organizational skills.
- *Level of teacher training:* All levels
- *School subject(s):* All subjects
- *Comments:* For ages over 5 years

5.4 Good practices — Inclusion

Good practice 42

This task can be used to teach students with disabilities about geometric shapes using Poly Universe and teaching fading techniques. This teaching technique implies that the student adopts the most important contents in smaller parts, by reading them aloud and supplementing sentences or connecting objects and concepts. In this case, the teacher prepares the boxes on which the names of the shapes from the Poly universe set are written and connects them with the shapes from the set. The student reads the name several times and points to the shape. After that, the teacher removed one of the cards and asked the student to name the shapes. When the student successfully completes this part of the task, the teacher removes the second card. Then he asked the student again to name the shapes. Finally, the teacher folds all the cards and asks the student to name the shapes.

Step 1. The teacher provides all information to the student and asks him to read it aloud and point to the shape.

Step 2**:** The teacher moves out one of the name cards and asks a student to name all shapes.

The teacher deletes one by one name cards and asks students to name the shape of figures. This process could be repeated several times till students get the knowledge to independently name the shape of figures.

- *Why this exercise is good:* This exercise provides practical activities to the students with disabilities and contributes to their knowledge in recognizing the shapes around them.
- *Level of teacher training:* Elementary school
- *School subject(s):* Math, language
- *Comments:* This exercise is dedicated to the students with cognitive disabilities, but with some modification could be usable for students with other disabilities as well.

Good practice 43

In order for students with disabilities to have the opportunity to practice combinatorics and orientation, teachers can use Poly-Universe and repeat the technique. Implementation of this

Teaching techniques require students to repeat the sequence given as a model. For the first few times, the student arranges the figures according to the model, and then – if the student's abilities allow it – without observing the model. When adapting the teaching content to the student, the teacher can use only pictures, and can use a combination of pictures and words (depending on the student's abilities). One example is given in the picture below.

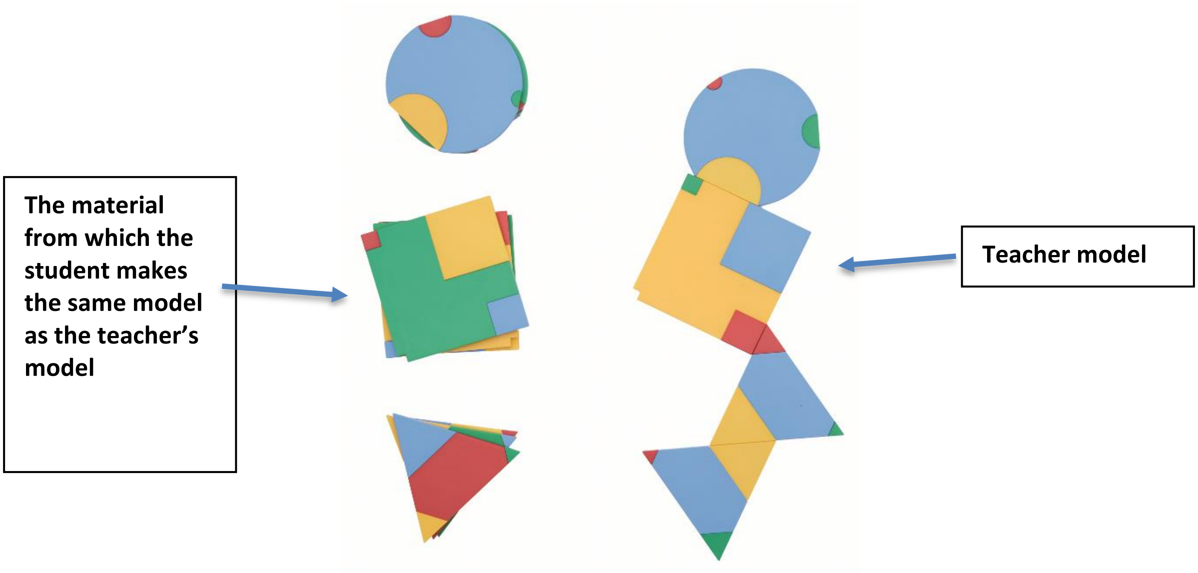
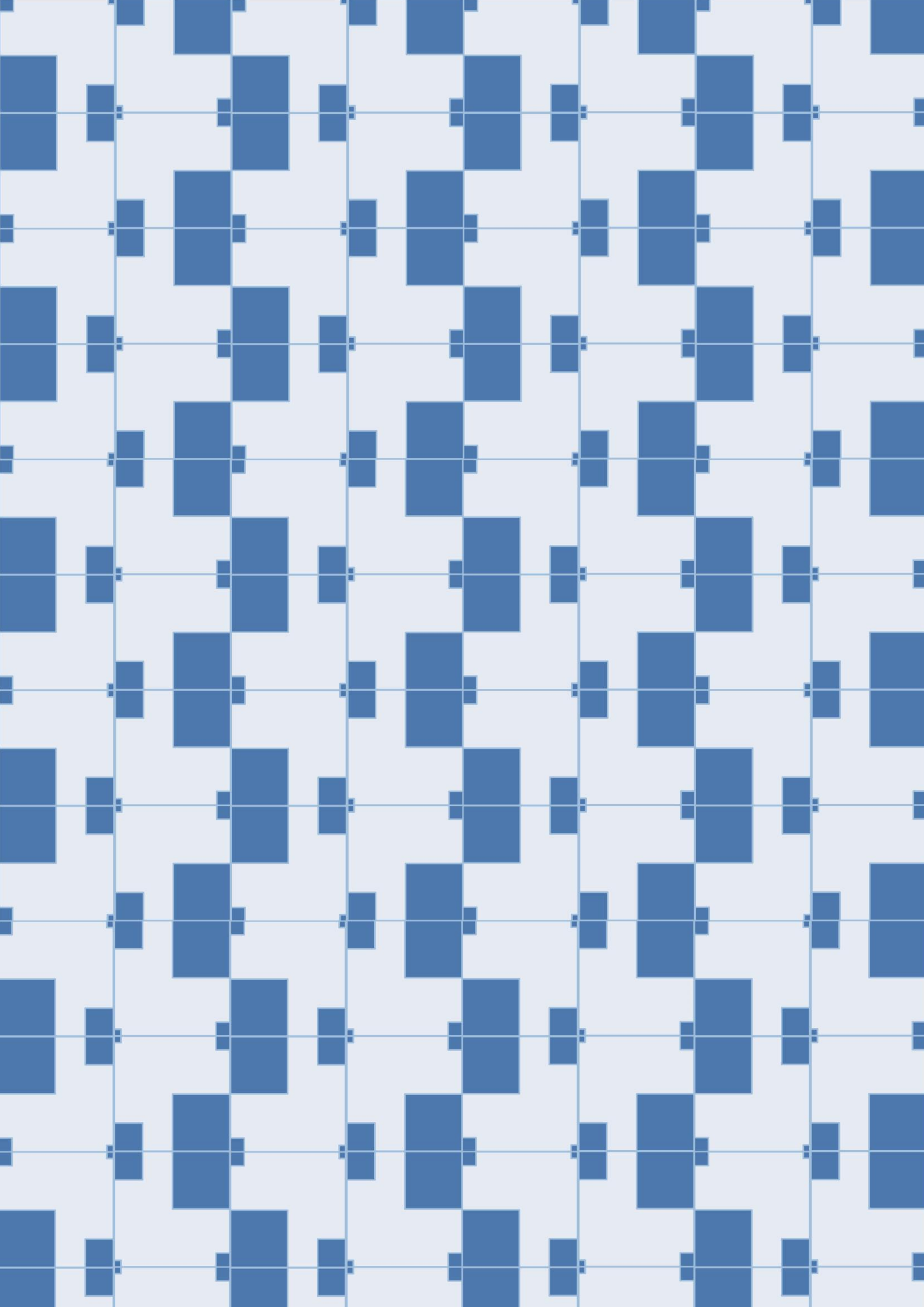


Figure 1

- *Why this exercise is good:* This exercise provides practical activities to the students with disabilities and contributes to their knowledge in recognizing the shapes around them, develops students' combinatorial abilities as *well as their ability to notice details.*
- *Level of teacher training:* Elementary school
- *School subject(s):* Math, art
- *Comments:* This exercise is dedicated to the students with cognitive disabilities, but with some modification could be usable for students with other disabilities as well. The teacher should always first create a simple model and only after that more complex in accordance with the abilities of the students.



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